Extending the Theory of k-Fibonacci and k-Lucas Numbers

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Author’s contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

The a:k:m-Fibonacci sequences $F_{a:k,m,n}$ and the a:k:m-Lucas sequences $L_{a:k,m,n}$ are introduced. The well-known k-Fibonacci and k-Lucas sequences become a particular case ($a=m=1$). One might be interested to meet the equally famous Jacobsthal sequence at $a=2, k=m=1$. Our brief results capture the most important properties relating to the assemblage mechanics of these sequences.

Keywords: a:k:m-Fibonacci numbers; a:k:m-Lucas numbers; Jacobsthal numbers; k-Fibonacci numbers; k-Lucas numbers; metallic means.

1 Introduction

The sequence of numbers

$$F_n = 1, 1, 2, 3, 5, 8, ...$$

(1.1)
well-known as the Fibonacci numbers in the literature [1-32], is arithmetically generated by the recurrence relation

\[ f_{n+2} = f_{n+1} + f_n, \quad n \geq 1 \]  
(1.2)

with initial conditions

\[ f_1 = f_2 = 1 \]  
(1.3)

The k-Fibonacci numbers introduced by Falcon and Plaza [1] are defined by

\[ F_{k,n} \left\{ \begin{align*} f_{k,n+2} &= k f_{k,n+1} + f_{k,n}, \quad n, k \geq 1 \\ f_{k,1} &= 1, f_{k,2} = k \end{align*} \right. \]  
(1.4)

In this communication we introduce the a:k:m-Fibonacci numbers defined by

\[ F_{a:k:m,n} \left\{ \begin{align*} f_{a:k:m,n+2} &= k f_{a:k:m,n+1} + a m f_{a:k:m,n}, \quad a, k, m, n \geq 1 \\ f_{a:k:m,1} &= 1, f_{a:k:m,2} = k \end{align*} \right. \]  
(1.5)

and the related a:k:m-Lucas numbers defined by

\[ L_{a:k:m,n} \left\{ \begin{align*} l_{a:k:m,n+2} &= k l_{a:k:m,n+1} + a m l_{a:k:m,n}, \quad a, k, m, n \geq 1 \\ l_{a:k:m,1} &= k, l_{a:k:m,2} = k^2 + 2am \end{align*} \right. \]  
(1.6)

based on the positive solution of the quadratic equation

\[ ax^2 - kx - m = 0 \]  
(1.7)

Let’s denote this solution \( \Omega_{a:k:m} \). We have that

\[ \Omega_{a:k:m} = \frac{k + \sqrt{k^2 + 4am}}{2a} \]  
(1.8)

Our very brief results are intended to show that the basic properties of the sequence (1.1) and the Lucas sequence are retained in all the a:k:m-Fibonacci and a:k:m-Lucas sequences respectively. One might be interested to learn that, for instance, the well-known Jacobsthal numbers

\[ J_{a:k:m,n} \left\{ \begin{align*} J_{a:k:m,n+2} &= k J_{a:k:m,n+1} + a m J_{a:k:m,n}, \quad a, k, m, n \geq 1 \\ J_{a:k:m,1} &= k, J_{a:k:m,2} = k^2 + 2am \end{align*} \right. \]  
(1.9)

are in fact employing the same concept as the Fibonacci, Pell, etc. numbers.

2 Results

Theorem 2.1

\[ a^{n-1}(\Omega_{a:k:m}^m)^n - f_{a:k:m,n} \Omega_{a:k:m}^m = m f_{a:k:m,n-1}, \quad n \geq 1 \]  
(2.1)

Proof

By induction. Base case: \( n = 1 \),

\[ \Omega_{a:k:m}^m - \Omega_{a:k:m} = 0 = m f_{a:k:m,0} \]
Inductive Hypothesis:
\[
a^{i-1}(\Omega^k_a)^i - f_{a:k,m,i} \Omega^k_a = mf_{a:k,m,i-1}, \quad i \geq 1
\]  
(2.2)

Inductive Conclusion:
\[
a^i(\Omega^k_a)^{i+1} - f_{a:k,m,i+1} \Omega^k_a = mf_{a:k,m,i} \quad i \geq 1
\]  
(2.3)

We have that
\[
a^i(\Omega^k_a)^{i+1} - f_{a:k,m,i+1} \Omega^k_a
\]
\[
= a^i(\Omega^k_a)^{i+1}(\Omega^k_a) - mf_{a:k,m,i} \Omega^k_a
\]
\[
= k (a^{i-1}(\Omega^k_a)^i - f_{a:k,m,i} \Omega^k_a) + \frac{k^2 + 4am}{2a} (a^i)(\Omega^k_a)^i - mf_{a:k,m,i} \Omega^k_a
\]
\[
= kmf_{a:k,m,i-1} + m(a^{i-1})(\Omega^k_a)^i - mf_{a:k,m,i} \Omega^k_a
\]
\[
= kmf_{a:k,m,i-1} + amf_{a:k,m,i-2}
\]
\[
= mf_{a:k,m,i}
\]

Induction is concluded, proof is complete.

**Theorem 2.2**

\[
f_{a:k,m,n+1} = af_{a:k,m,n} \Omega^k_a + \left(\frac{m}{\Omega^k_a}\right)^n, \quad n \geq 1
\]  
(2.4)

**Proof**

By induction. Base case: \( n = 1 \),
\[
a \Omega^k_a + \left(\frac{m}{\Omega^k_a}\right)^1 = k = f_{a:k,m,2}
\]

Inductive Hypothesis:
\[
f_{a:k,m,i+1} = af_{a:k,m,i} \Omega^k_a + \left(\frac{m}{\Omega^k_a}\right)^i, \quad i \geq 1
\]  
(2.5)

Inductive Conclusion: We prove that
\[
f_{a:k,m,i+2} = af_{a:k,m,i+1} \Omega^k_a + \left(\frac{m}{\Omega^k_a}\right)^{i+1}, \quad i \geq 1
\]  
(2.6)

We obtain
\[
af_{a:k,m,i+1} \Omega^k_a + \left(\frac{m}{\Omega^k_a}\right)^{i+1}
\]
\[
= k (af_{a:k,m,i} \Omega^k_a + \left(\frac{m}{\Omega^k_a}\right)^i) - k + \frac{k^2 + 4am}{2a} \left(\frac{m}{\Omega^k_a}\right)^i + amf_{a:k,m,i-1} \Omega^k_a
\]
\[
= kmf_{a:k,m,i+1} + amf_{a:k,m,i} \Omega^k_a + \left(\frac{m}{\Omega^k_a}\right)^{i+1}
\]
\[
= kmf_{a:k,m,i+1} + amf_{a:k,m,i}
\]
\[
= f_{a:k,m,i+2}
\]
Having concluded the induction process, proposition is true.

**Theorem 2.3**

\[ l_{a:k:m,n} = a^n(\Omega_a^{k,m})^n + \left(\frac{-m}{\Omega_a^{k,m}}\right)^n, \quad n \geq 1 \quad (2.7) \]

**Derivation**

Notice that, by definition,

\[ l_{a:k:m,n} = amf_{a:k:m,n-1} + f_{a:k:m,n+1} \quad (2.8) \]

From proved equations (2.1) and (2.4) this becomes

\[
\begin{align*}
& a^n(\Omega_a^{k,m})^n - f_{a:k:m,n} \Omega_a^{k,m} + amf_{a:k:m,n} \Omega_a^{k,m} + \left(\frac{-m}{\Omega_a^{k,m}}\right)^n \\
& = a^n(\Omega_a^{k,m})^n + \left(\frac{-m}{\Omega_a^{k,m}}\right)^n
\end{align*}
\]

Theorems 2.1 to 2.3 capture the basic properties of a:k:m-Fibonacci and a:k:m-Lucas sequences relating to assembly mechanics. Without further proof we state Catalan’s and d’Ocagne’s identities respectively:

\[
\begin{align*}
& f_{a:k:m,n}^2 - f_{a:k:m,n+r} f_{a:k:m,n-r} = (-1)^{n-r}(am)^{n-r} f_{a:k:m,r}^2, \quad n, r \geq 1 \quad (2.9) \\
& f_{a:k:m,n+1} f_{a:k:m,n+1} - f_{a:k:m,n} f_{a:k:m,r+1} = (-1)^n(am)^n f_{a:k:m,r-n}, \quad n, r \geq 1 \quad (2.10)
\end{align*}
\]

### 3 Conclusion

The a:k:m-Fibonacci and a:k:m-Lucas sequences extend not only the theory of k-Fibonacci and k-Lucas numbers but of metallic means [10] also. That we have shown that the Jacobsthal numbers [19] for example employ the same concept as the classic Fibonacci numbers goes a long way in the unification of seemingly disparate ideas and opening new avenues of research.

**Competing Interests**

Author has declared that no competing interests exist.

**References**


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