A New Criterion for Borel-Euler Summability Method of Fourier Series

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

This paper introduces a new theorem on \((B)\) product summability of Fourier series which is the generalization of the result given by Izumi S. [1] under analogous conditions.

Keywords: \((E,1)\) summability; \((C,1)(E,1)\) summability; Borel summability; \((B)(E,1)\) summability.


1 Introduction

Let \(f(x)\) be a function integrable in the sense of Lebesgue over the interval \((-\pi, \pi)\) and periodic with the period \(2\pi\). Titchmarsh [2]. Let the Fourier series associated with \(f(x)\) be

\[
\sum_{n=0}^{\infty} A_n(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]  

(1.1)

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An infinite series \( \sum u_n \) with partial sums \( S_n \) are said to be summable \( (B)(E,1) \) to \( s \), if

\[
\frac{1}{n} \sum_{m=1}^{n} E_m \to s \quad \text{as} \quad n \to \infty
\]  

(1.2)

Where \( E_m \) stands for the \( (E,1) \) mean of \( S_n \), Hardy, [3].

We shall use the fixed real numbers \( x \) and \( s \), the following notations,

\[
\phi(t) = f(x+t) + f(x-t) - 2s
\]

\[
\Phi(t) = \int_0^t \phi(u) du
\]

\[
h(n, t) = \frac{1}{t} \sin \left( \frac{p \sin t + t}{2} \right) \exp \left( \frac{p \left( 1 - \cos t \right)}{2} \right)
\]

\[
h_1(n, t) = \frac{d}{dt} \left[ \sin \left( \frac{p \sin t + t}{2} \right) \exp \left( \frac{p \left( 1 - \cos t \right)}{2} \right) \right]
\]

In 1903, Lebesgue H. [4] gave the convergence criteria for Fourier series, at a point \( x \) by proving the result.

**THEOREM A:** If

\[
\int_0^t \phi(u) du = o(t) \quad \text{as} \quad t \to 0
\]

and

\[
\int_{\frac{1}{n}}^{\frac{2}{n}} \frac{\phi(t) - \phi(t + \frac{1}{n})}{t} dt = o(1) \quad \text{as} \quad n \to \infty
\]

then the Fourier series of \( f(x) \) converges to \( f(x) \) at the point \( x \).

Generalising Theorem A for \( (C,1) \) summability of Fourier series, Izumi S. [1] proved the following result.

**THEOREM B:** If

\[
\int_0^t \phi(u) du = o(t) \quad \text{as} \quad t \to 0
\]

and
\[ \int_{\eta}^{\pi} \frac{\phi(t + \frac{x}{n}) - \phi(t)}{t^2} \, dt = o(1) \quad \text{as} \quad n \to \infty \]

then the Fourier series is summable \((C, 1)\) to the sum \(s\).

Recently, Saxena K. [5-6] has proved a theorem for the product summability \((C, 1)(E, 1)\) of Fourier series under analogous conditions.

**THEOREM C:** If

\[ \int_{0}^{\eta} \phi(u) \, du = o(t) \quad \text{as} \quad t \to 0 \]

and

\[ \int_{0}^{\eta} \frac{\phi(t) - \phi(t + \frac{x}{n})}{t^2} \cos^n \frac{t}{2} \, dt = o(n) \]

as \(n \to \infty\), for fixed positive number \(\eta > 0\), then the Fourier series \((1.1)\) is \((C, 1)(E, 1)\) summable to \(s\) at the point \(x\).

Various researchers [7-12] proved some interesting results on summability of Fourier series.

Since, under the conditions of Theorem B, Fourier series is not \((E, 1)\) summable to any fixed number, so it is natural to expect the extension of Theorem B for the product summability \((B)(E, 1)\) of the Fourier series under analogous conditions.

**2 Main Results**

In this section, we prove the result by generalizing the conditions of Theorem B for Borel-Euler product summability of Fourier series.

**THEOREM 2.1:** If

\[ \int_{0}^{\eta} \phi(u) \, du = o(t) \quad \text{as} \quad t \to 0 \quad (2.1) \]

and

\[ \int_{0}^{\eta} \frac{\phi(t) - \phi(t + \frac{x}{n})}{t} \exp \left\{ - \frac{1}{p} \left( \frac{1 - \cos t}{2} \right) \right\} \, dt = o(1) \quad (2.2) \]

As \(p \to \infty\), for some fixed positive number \(\eta > 0\), then the Fourier series \((1.1)\) is \((B)(E, 1)\) summable to \(s\) at the point \(x\).
3 Relations

\[ \sum_{k=0}^{m} \binom{m}{k} \sin \left( k + \frac{1}{2} \right) t = 2^m \cos^m \frac{t}{2} \sin \left( \frac{m+1}{2} \right) t \]  

(3.1)

Proof:

\[ \sum_{k=0}^{m} \binom{m}{k} \sin \left( k + \frac{1}{2} \right) t = \text{Im} \sum_{k=0}^{m} \binom{m}{k} \exp \left\{ i \left( k + \frac{1}{2} \right) t \right\} \]

\[ = \text{Im} \exp \left\{ \frac{it}{2} \right\} \left\{ 1 + \exp(it) \right\}^m \]

\[ = 2^m \cos^m \frac{t}{2} \sin \left( \frac{m+1}{2} \right) t \]

Also,

\[ \exp(-p) \sum_{m=0}^{\infty} \frac{p^m}{m!} \cos^m \frac{t}{2} \sin \left( \frac{m+1}{2} \right) t = \exp \left\{ -p \left( \frac{1 - \cos t}{2} \right) \sin \left( \frac{p \sin t + t}{2} \right) \right\} \]  

(3.2)

Proof:

\[ \exp(-p) \sum_{m=0}^{\infty} \frac{p^m}{m!} \cos^m \frac{t}{2} \sin \left( \frac{m+1}{2} \right) t = \exp(-p) \text{Im} \sum_{m=0}^{\infty} \frac{p^m}{m!} \cos^m \frac{t}{2} \exp \left\{ i \left( \frac{m+1}{2} \right) t \right\} \]

\[ = \exp(-p) \text{Im} \exp \left\{ \frac{it}{2} \right\} \sum_{m=0}^{\infty} \frac{p^m}{m!} \cos^m \frac{t}{2} \exp \left\{ \frac{int}{2} \right\} \]

\[ = \exp(-p) \text{Im} \exp \left\{ \frac{it}{2} \right\} \exp \left\{ p \cos^2 \frac{t}{2} \exp \left\{ \frac{it}{2} \right\} \right\} \]

\[ = \exp(-p) \exp \left\{ p \cos^2 \frac{t}{2} \right\} \sin \left( \frac{p \sin t + t}{2} \right) \]

\[ = \exp \left\{ -p \left( \frac{1 - \cos t}{2} \right) \right\} \sin \left( \frac{p \sin t + t}{2} \right) \]

4 The Estimates

We shall require the following estimates, the first may be verified easily

For \( 0 < t < \frac{\pi}{p} \)

\[ h(n, t) = O(p + 1) \]  

(4.1)
\[ \frac{d}{dt} h(n, t) = O \left( \frac{p + 1}{2} \right) \frac{1}{t} \]  
(4.2)

For \( t > \frac{\pi}{p} \)

\[ \exp \left\{ -p \sin^2 \left( \frac{t}{2} \right) \right\} - \exp \left\{ -p \sin^2 \left( \frac{p}{2} \right) \right\} = O(t) \]  
(4.3)

\[ \frac{d}{dt} \exp \left\{ -p \sin^2 \left( \frac{t}{2} \right) \right\} - \exp \left\{ -p \sin^2 \left( \frac{p}{2} \right) \right\} = 0 \]  
(4.4)

\[ \left( \frac{1}{t} - \frac{1}{t + \frac{\pi}{p}} \right) = o \left( \frac{1}{pt^2} \right) \]  
(4.5)

\[ \frac{d}{dt} \left( \frac{1}{t} - \frac{1}{t + \frac{\pi}{p}} \right) = o \left( \frac{1}{pt^3} \right) \]  
(4.6)

\[ \frac{d}{dt} \left( \frac{1}{t} - \frac{1}{t + \frac{\pi}{p}} \right) \exp \left\{ -p \left( \frac{1 - \cos t}{2} \right) \sin \frac{pt}{2} \right\} = o \left( \frac{1}{t^2} \right) \]  
(4.7)

5 Required Lemmas

**Lemma 5.1:**

\[ \int_{\frac{\pi}{p}}^{\frac{\pi}{p}} \frac{\phi(t)}{t} \sin \left( p \sin t + t \right) - \sin pt \exp \left\{ \frac{p(1 - \cos t)}{2} \right\} dt = o(1) \]

**Proof:**

Using second mean value theorem and integrating by parts, we have
= \int_{\pi/p}^{(\pi/p)^\rho} \frac{\phi(t)}{t} \sin\left(p\sin t + t\right) - \sin pt \exp\left[p\sin^2 \frac{t}{2}\right] dt \\
= \frac{1}{\exp\left(p\sin^2 \frac{t}{2}\right)} \left[\int_{\pi/p}^{(\pi/p)^\rho} \frac{\phi(t)}{t} \left\{\sin(p\sin t + t) - \sin pt\right\} dt\right] \\

Since \(\frac{1}{3} \leq \alpha \leq \beta < 1\)

= \mathcal{O}(1) \int_{\pi/p}^{(\pi/p)^\rho} \frac{\phi(t)}{t} \left.o\right|^2 d\theta \\
= \mathcal{O}(p) \int_{\pi/p}^{(\pi/p)^\rho} \phi(t) d\theta \\
= \mathcal{O}(p) \left[\theta^2 - 2 \int \theta d\theta \left.\right|_{\pi/p}^{\pi/p} \right] \\
= \mathcal{O}(p) \left[\theta^2 \left.\right|_{\pi/p}^{\pi/p} \right] \\
= \mathcal{O}(1)\), as \(p \to \infty\)

Lemma 5.2:

\[ \int_{\pi/p}^{(\pi/p)^\rho} \frac{\phi(t + \pi/p)}{t} \left\{\exp\left[-p(1 - \cos t)\right] - \exp\left[-p(1 - \cos(t + \pi/p))\right]\right\} \sin pt dt = \mathcal{O}(1) \]

Proof:

By mean value theorem of differential calculus,

\[ \frac{\exp\left[-p(1 - \cos(t + \pi/p))\right] - \exp\left[-p(1 - \cos t)\right]}{t + \frac{\pi}{p} - t} = \frac{d}{dt} \exp\left[-p(1 - \cos \theta)\right] \quad \text{where} \quad \theta = t + \frac{l\pi}{p}; \quad (0 < l < 1) \]

\[ = -p \sin \theta \exp\left[-p(1 - \cos \theta)\right] \]

\[ = -p \sin(t + l\pi/p) \exp\left[-p(1 - \cos(t + l\pi/p))\right] \]

Consequently,

\[ \exp\left[-p(1 - \cos t)\right] - \exp\left[-p(1 - \cos(t + \pi/p))\right] = \pi \sin(t + l\pi/p) \exp\left[-p(1 - \cos(t + l\pi/p))\right] = \mathcal{O}(1) \]
Therefore,
\[
\int_{\pi/p}^{(\pi/p)^p} \frac{\phi(t + \pi/p)}{t} \sin(t + l\pi/p) \exp[- p(1 - \cos(t + l\pi/p))] \sin pt \, dt
\]
\[
= o(1) \int_{\pi/p}^{(\pi/p)^p} |\phi(t + \pi/p)| \, dt
\]
\[
= o(1), \text{ as } p \to \infty \quad \text{By (2.1)}
\]

**Lemma 5.3:**
\[
\int_{\pi/p}^{(\pi/p)^p} \phi(t + \pi/p) \left[1 - \frac{1}{t^2} \right] \exp[- p(1 - \cos t)] \sin pt \, dt = o(1)
\]

**Proof:**
By (2.1) and integrating by parts
\[
\leq \frac{\pi}{p} \int_{\pi/p}^{(\pi/p)^p} \left|\phi(t + \pi/p)\right| \frac{1}{t(t + \pi/p)} \, dt
\]
\[
< \frac{\pi}{p} \int_{\pi/p}^{(\pi/p)^p} \left|\phi(t + \pi/p)\right| \frac{1}{t^2} \, dt
\]
\[
= o\left(\frac{1}{p}\right) \int_{\pi/p}^{(\pi/p)^p} o\left(\frac{1}{t^2}\right) + 2 \int\left[\frac{o(t + \pi/p)}{t^3} \right]_{\pi/p}^{(\pi/p)^p} \, dt
\]
\[
= o\left(\frac{1}{p}\right) \int_{\pi/p}^{(\pi/p)^p} + 2 \int\left[\frac{(\pi/p)^p}{t^2}\right] \, dt
\]
\[
= o(1), \text{ as } p \to \infty
\]

**Lemma 5.4:**
\[
\int_{0}^{(\pi/p)^p} \frac{\phi(t + \pi/p)}{t + \pi/p} \exp[- p(1 - \cos(t + \pi/p))] \sin pt \, dt = o(1)
\]

**Proof:**
By (2.1) and change of variables,
\[
\leq \int_{\pi/p}^{(2\pi/p)} \left|\phi(t)\right| \, dt
\]
\[
= o\left(\frac{m}{\pi}\right) \int_{2\pi/p}^{(2\pi/p)^p} \left|\phi(t)\right| \, dt
\]
\[
= o(1), \text{ as } p \to \infty
\]
By using the above-proved lemmas, we will prove the main theorem.

6 Proof of the Main Theorem

Let $S_n$ be the $n^{th}$ partial sum of the Fourier series (1.1), following Zygmund [13], we have

$$S_n - s = \frac{1}{2\pi} \int_0^\pi \phi(t) \frac{\sin(\frac{\pi}{2}n + \frac{1}{2}t)}{\sin \frac{\pi}{2} t} \, dt$$

$(E,1)$ transform of $S_n$ is denoted by $E_n$, we have in relation (3.1)

$$E_n = \frac{1}{2\pi} \int_0^\pi \phi(t) \sum_{k=0}^n \sin \left( \frac{\pi}{2}n + \frac{1}{2}t \right) \, dt$$

$$= \frac{1}{2\pi} \int_0^\pi \phi(t) \cos^{n+1} \frac{t}{2} \sin \left( \frac{m+1}{2} \right) \, dt$$

(6.1)

Superimposing Borel transform on $E_n$, hence by relation (3.2), we find

$$B_p = \frac{1}{2\pi} e^{-p} \int_0^\pi \phi(t) \sum_{m=0}^\infty \frac{p^m}{m!} \cos^m \frac{t}{2} \sin \left( \frac{m+1}{2} \right) \, dt$$

$$= \frac{1}{2\pi} \int_0^\pi \phi(t) \exp \left\{ - p \left( \frac{1 - \cos t}{2} \right) \right\} \sin \left( \frac{p \sin t + t}{2} \right) \, dt$$

which may also be written as

$$B_p = \frac{1}{2\pi} \int_0^\pi \phi(t) \exp \left\{ - p \left( \frac{1 - \cos t}{2} \right) \right\} \sin \left( \frac{p \sin t + t}{2} \right) \, dt$$

$$= \frac{1}{2\pi} \int_0^\pi \phi(t) \exp \left\{ - p \left( \frac{1 - \cos t}{2} \right) \right\} \sin \left( \frac{p \sin t + t}{2} \right) \, dt + o(1)$$

$$= \int \left[ I + I_2 + I_3 \right] + o(1), \text{ (say)}$$

Integrating by parts and using estimates (4.1), (4.2) given conditions
Using Lemma

\[ I_1 = \frac{1}{\pi} \int_{0}^{\pi/p} \frac{\varphi(t)}{t} \exp\left\{- p \left( \frac{1 - \cos t}{2} \right) \right\} \sin \left( \frac{p \sin t + t}{2} \right) dt \]

\[ = \frac{1}{\pi} \int_{0}^{\pi/p} \varphi(t) h(n,t) dt \]

\[ = \frac{1}{\pi} \int_{0}^{\pi/p} \Phi_1(t) h(n,t) - \int_{0}^{\pi/p} \frac{d}{dt} h(n,t) \Phi_1(t) dt \]

\[ = \frac{1}{\pi} \int_{0}^{\pi/p} \left\{ o(t) O(p) \right\} dt \]

\[ = o(1) + o(p) \]

\[ = o(1) \quad \text{as} \quad p \to \infty \]

By given condition, we get

\[ 2I_2 = \frac{1}{\pi} \int_{\pi/p}^{\pi/p} \frac{\varphi(t)}{t} \exp\left\{- p \left(1 - \cos t\right)\right\} \sin pt dt \]

\[ - \frac{1}{\pi} \int_{0}^{\pi/p} \frac{\varphi(t + \pi/p)}{t + \pi/p} \exp\left\{- p \left(1 - \cos t\right)\right\} \sin pt dt \]

\[ = \frac{1}{\pi} \int_{\pi/p}^{\pi/p} \varphi(t) - \varphi(t + \pi/p) \exp\left\{- p \left(1 - \cos t\right)\right\} \sin pt dt \]

\[ + \frac{1}{\pi} \int_{\pi/p}^{\pi/p} \varphi(t + \pi/p) \left[ \frac{1}{t + \pi/p} - \frac{1}{t + \pi/p} \right] \exp\left\{- p \left(1 - \cos t\right)\right\} \sin pt dt \]

\[ - \frac{1}{\pi} \int_{0}^{\pi/p} \varphi(t + \pi/p) \exp\left\{- p \left(1 - \cos t + \pi/p\right)\right\} \sin pt dt \]

\[ + \frac{1}{\pi} \int_{\pi/p}^{\pi/p} \varphi(t + \pi/p) \exp\left\{- p \left(1 - \cos t + \pi/p\right)\right\} \sin pt dt \]

\[ = e_1 + e_2 + e_3 + e_4 + e_5 \]

By given condition, we get

\[ |e_i| \leq \frac{1}{\pi} \int_{\pi/p}^{\pi/p} \left| \frac{\varphi(t) - \varphi(t + \pi/p)}{t} \right| \exp\left\{- p \left(1 - \cos t\right)\right\} dt \]

\[ = o(1) \quad \text{as} \quad p \to \infty \]
Applying Lemma (5.2), (5.3) and (5.4), we get

\[ e_3 = e_4 = e_5 = o(1) \]

Lastly considering \( e_5 \), we have by change of variables

\[
e_5 = \frac{1}{\pi} \int_{(\pi/p)^r}^{(\pi/p)^r} \frac{\phi(t)}{t} \exp\{-p(1 - \cos t)\} \sin pt \, dt
\]

\[
|e_5| \leq \frac{1}{\pi} \int_{(\pi/p)^r}^{(\pi/p)^r} \phi(t) \, dt
\]

\[
\leq \frac{1}{\pi} \left( \frac{p}{\pi} \right)^{\alpha} \int_{(\pi/p)^r}^{(\pi/p)^r} \phi(t) \, dt
\]

\[
= O\left( \frac{1}{p^{-\alpha}} \right)
\]

\[
= o(1) \quad \text{as } p \to \infty
\]

By the continuity parts of \( \int |\phi(t)| \, dt \) and the function \( \alpha < \frac{1}{2} \)

Thus, \( I_2 = o(1) \)

Now, at last, we have

\[
I_3 = \frac{1}{\pi} \int_{(\pi/p)^r}^{\delta} \frac{\phi(t)}{t} \exp\left\{-p\left(1 - \frac{\cos t}{2}\right)\right\} \sin\left(\frac{p \sin t + t}{2}\right) \, dt
\]

\[
= \frac{1}{\pi} \int_{(\pi/p)^r}^{\delta} \frac{\phi(t) \sin(p \sin t + t)}{t} \exp\left\{p \sin^2 \frac{t}{2}\right\} \, dt
\]

\[
\leq \frac{1}{\pi} \exp\left\{p \sin^2 \frac{\delta}{2}\right\} \int_{(\pi/p)^r}^{\delta} \phi(t) \sin(p \sin t + t) \, dt
\]

\[
\leq \frac{p^\alpha}{\exp[p^{-2\alpha}]} \int_{(\pi/p)^r}^{\delta} |\phi(t)| \, dt
\]

\[
= o(1) \quad \text{as } p \to \infty
\]
This completes the proof of theorem 2.1.

7 Conclusion

In this paper, we have introduced the product summability of Fourier series using Borel-Euler summation method. The present theorem extends, generalizes and improves many existing results on summability of Fourier series and its allied series. This result may be a motivation to other researchers to carry out the outcomes in the field of summability theory.

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Competing Interests

Authors have declared that no competing interests exist.

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