The Solitary Travelling Wave Solutions of Some Nonlinear Partial Differential Equations Using the Modified Extended Tanh Function Method with Riccati Equation

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Authors’ contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this work, we aimed to construct a variety of solitary travelling wave solutions of a wide class of nonlinear partial differential equations (PDE’s) that is governed by a presented single nonlinear partial differential equation (PDE) using the powerful modified extended Tanh method with Riccati equation. More general solutions are successfully constructed including the previous known formal solutions such as shock wave, periodic and weirstrass solutions.

Keywords: Modified Extended Tanh; Riccati equation; solitary wave solutions; nonlinear PDE’s.

1 Introduction

Nonlinear partial differential equations (PDE’s) have a major importance in the field of nonlinear physical phenomena such as plasma physics, optical fibers, solid state physics, fluid mechanics and other branches of science, most of these equations are nonlinear and difficult to handle due to the nonlinearity term(s). In

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recent years, the investigation of the solitary travelling wave solutions of nonlinear PDE's has become more and more attractive due to the availability of computer symbolic systems like Mathematica or Maple which allow us to perform some complicated, tedious calculations and computations on a computer. They help us to find new exact solutions of nonlinear PDE's [1,2]. Therefore, a variety of effective straightforward methods for obtaining and constructing general formal solutions of nonlinear PDE's have been proposed and studied by many authors such as Darboux transformation, Cole-Hopf transformation, Hirota method, Painlevé method [3,4,5,6,7], Jacobi elliptic function method [8], sine-cosine methods [9,10].

In particular, the Modified Extended Tanh-function method [11,12] is widely recognized as one of the most powerful tools used recently in a favor of searching for explicit solitary travelling wave solutions PDE's. The method has been developed over years, the extended Tanh function method [12,13,14,15,16,17], modified Tanh method [17], Tanh-coth method [18], generalized hyperbolic–function method –see [19].

In this presented work; we aim to employ the modified extended Tanh function method with Riccati equation –see [11,12] –for solving some nonlinear PDE’s which are classified as the following governing general nonlinear PDE:

\[ u_t + \alpha uu_x + \beta u^2 u_x + \nu u_{xx} + \mu u_{xx} = 0 \]  

(1)

Giving the following well-known nonlinear PDE’s:

The modified Korteweg–de Vries (mKdV) equation (with \( \alpha = 0, \nu = 0 \)):

\[ u_t + \beta u^2 u_x + \mu u_{3x} = 0 \]  

(2)

The modified Korteweg–de Vries Burgers (mKdVB) equation (with \( \alpha = 0, p = 3 \)):

\[ u_t + \beta u^2 u_x + \nu u_{xx} + \mu u_{3x} = 0 \]  

(3)

The modified Kawahara (mK) equation (with \( \alpha = 0, p = 3, 5 \)):

\[ u_t + \beta u^2 u_x + \mu u_{3x} - \delta u_{5x} = 0 \]  

(4)

According to the methodology of the used method, a nonlinear PDE in two variables \( x, t \) takes the form:

\[ \Phi(u, u_x, u_{xx}, \ldots) = 0 \]  

(5)

Using the wave transformation \( u(x,t) = U(\xi) \) with a wave variable of the form \( \xi = x - \omega t \), then (5) becomes the ODE:

\[ \Phi_1(U, U', U'', \ldots) = 0 \]  

(6)

which is integrated as long as all terms contain derivatives. Introducing a new variable \( \phi = \phi(\xi) \), that satisfies the Riccati equation of the form:

\[ \frac{d\phi}{d\xi} = k + \phi^2 \]  

(7)
where $k$ is a real constant. The modified extended tanh function method admits the finite series expansion:

$$u(x,t) = U(\xi) = \sum_{i=0}^{m} a_i \phi^i + \sum_{i=1}^{m} b_i \phi^{-i}$$  \hspace{1cm} (8)

where $m$ is a positive integer to be determined by balancing the highest-order linear term with the nonlinear term in (6). Substituting (7) along with (8) into (6), setting to zero all coefficients of $\phi^i$, a nonlinear algebraic system is generated with parameters $a_0, a_1, b_1, k, \omega$. The Riccati equation (7) has the general solution:

$$\phi(\xi) = \begin{cases} 
-\sqrt{-k} \tanh\left(\sqrt{-k} (x - \omega t)\right), & k < 0 \\
-\sqrt{-k} \coth\left(\sqrt{-k} (x - \omega t)\right), & k < 0 \\
-\frac{1}{\xi}, & k = 0 \\
\sqrt{k} \tan\left(\sqrt{k} (x - \omega t)\right), & k > 0 \\
-\sqrt{k} \cot\left(\sqrt{k} (x - \omega t)\right), & k > 0
\end{cases}$$ \hspace{1cm} (9)

Therefore, by the sign test of $k$ the exact solutions of (5) are obtained.

2 The Solitary Travelling Wave Solution of the mKdV (2), mKdVB (3) and mKawahara (4)

2.1 The Modified Korteweg –de Vries (mKdV) equation

According to the used method, the mKdV equation becomes the ODE:

$$-\omega U' + \beta U^2U' + \mu U''' = 0$$ \hspace{1cm} (10)

Integrating (10) with zero constant of integration to become:

$$-\omega U + \frac{\beta}{3} U^3 + \mu U''' = 0$$ \hspace{1cm} (11)

with $m = 1$ (by balancing $U'''$ with $U^3$), the finite expansion (8) would have the form:

$$U(\xi) = a_0 + a_1 \phi(\xi) + b_1 \phi^{-1}(\xi)$$ \hspace{1cm} (12)

Substituting (12) into (11) and with the use of (7), the following algebraic system is obtained for the coefficients of $\phi^i, i = 0, \pm 1, \pm 2, \pm 3,$: 
The system in (13) is solved to give the following sets of solutions, and by the sign test of $k$ the exact solutions of mKdV equation are obtained as follows:

**Case 1.**

$$k = -\frac{\omega}{4\mu}, \quad a_1 = \mp \sqrt{\frac{6\mu}{\beta}}, \quad b_1 = \pm \frac{3}{2\beta\mu} \omega, \quad a_0 = 0$$  

(14)

where $\omega$ is an arbitrary constant.

If $\frac{\omega}{\mu} > 0$, then $k < 0$, substituting (14) into (12) and by the use of (9), the solution of mKdV is given by:

$$u_1(x,t) = \pm \sqrt{\frac{3\omega}{2\beta}} \Tanh\left[\frac{\omega}{2\mu}(x - t\omega)\right]$$

$$= \mp \frac{\omega}{4\mu} \sqrt{\frac{3\omega}{2\beta}} \Coth\left[\frac{\omega}{2\mu}(x - t\omega)\right]$$  

(15)

If $\frac{\omega}{\mu} < 0$, then $k > 0$, substituting (14) into (12) and by the use of (9), the solution of mKdV is given by:

$$u_2(x,t) = \mp \sqrt{\frac{3\omega}{2\beta}} \Tan\left[\frac{\omega}{2\mu}(x - t\omega)\right]$$

$$= \mp \frac{\omega}{4\mu} \sqrt{\frac{3\omega}{2\beta}} \Cot\left[\frac{1-\omega}{2\mu}(x - t\omega)\right]$$  

(16)

**Case 2.**

$$k = \frac{\omega}{8\mu}, \quad a_1 = \mp \sqrt{\frac{6\mu}{\beta}}, \quad b_1 = \pm \frac{\omega}{4\mu} \sqrt{\frac{3}{2\beta\mu}}, \quad a_0 = 0$$  

(17)

where $\omega$ is an arbitrary constant.
If \( \frac{\omega}{\mu} < 0 \), then \( k < 0 \), substituting (17) into (12) and by the use of (9), the solution of mKdV is given by:

\[
\begin{align*}
    u_3(x,t) &= \pm \frac{\tilde{a}\sqrt{3} \sqrt{\mu}}{2 \sqrt{\beta}} \frac{1}{\sqrt{\mu}} \sqrt{\frac{\omega}{\mu} (x - t\omega)} \\
    \tilde{a} &= \frac{3 \omega}{2 \sqrt{\beta} \sqrt{\mu}} \\
    u_4(x,t) &= \pm \frac{\tilde{a}\sqrt{3} \sqrt{\mu}}{16 \sqrt{\beta} \sqrt{\mu}} \frac{1}{\sqrt{\mu}} \sqrt{\frac{\omega}{\mu} (x - t\omega)} \\
    a_0 &= a_1 = 0
\end{align*}
\]

(18)

If \( \frac{\omega}{\mu} > 0 \), then \( k > 0 \), substituting (17) into (12) and by the use of (9), the solution of mKdV is given by:

\[
\begin{align*}
    u_5(x,t) &= \pm \frac{\tilde{a}\sqrt{3} \sqrt{\mu}}{2 \sqrt{\beta}} \frac{1}{\sqrt{\mu}} \sqrt{\frac{\omega}{\mu} (x - t\omega)} \\
    \tilde{a} &= \frac{3 \omega}{2 \sqrt{\beta} \sqrt{\mu}} \\
    u_6(x,t) &= \pm \frac{\tilde{a}\sqrt{3} \sqrt{\mu}}{2 \sqrt{\beta} \sqrt{\mu}} \frac{1}{\sqrt{\mu}} \sqrt{\frac{\omega}{\mu} (x - t\omega)} \\
    a_0 &= a_1 = 0
\end{align*}
\]

(19)

Case 3.

\[
k = \frac{\omega}{2 \mu}, \quad b_1 = -\frac{\tilde{a}\sqrt{3} \sqrt{\mu}}{\beta}, \quad a_0 = a_1 = 0
\]

(20)

where \( \omega \) is an arbitrary constant.

If \( \frac{\omega}{\mu} < 0 \), then \( k < 0 \), substituting (20) into (12) and by the use of (9), the solution of mKdV is given by:

\[
\begin{align*}
    u_3(x,t) &= \pm \frac{\tilde{a}\sqrt{3} \sqrt{\mu}}{2 \sqrt{\beta}} \frac{1}{\sqrt{\mu}} \sqrt{\frac{\omega}{\mu} (x - t\omega)} \\
    \tilde{a} &= \frac{3 \omega}{2 \sqrt{\beta} \sqrt{\mu}} \\
    u_4(x,t) &= \pm \frac{\tilde{a}\sqrt{3} \sqrt{\mu}}{16 \sqrt{\beta} \sqrt{\mu}} \frac{1}{\sqrt{\mu}} \sqrt{\frac{\omega}{\mu} (x - t\omega)} \\
    a_0 &= a_1 = 0
\end{align*}
\]

(21)

If \( \frac{\omega}{\mu} > 0 \), then \( k > 0 \), substituting (20) into (12) and by the use of (9), the solution of mKdV is given by:

\[
\begin{align*}
    u_5(x,t) &= \pm \frac{\tilde{a}\sqrt{3} \sqrt{\mu}}{2 \sqrt{\beta} \sqrt{\mu}} \frac{1}{\sqrt{\mu}} \sqrt{\frac{\omega}{\mu} (x - t\omega)} \\
    \tilde{a} &= \frac{3 \omega}{2 \sqrt{\beta} \sqrt{\mu}} \\
    u_6(x,t) &= \pm \frac{\tilde{a}\sqrt{3} \sqrt{\mu}}{2 \sqrt{\beta} \sqrt{\mu}} \frac{1}{\sqrt{\mu}} \sqrt{\frac{\omega}{\mu} (x - t\omega)} \\
    a_0 &= a_1 = 0
\end{align*}
\]

(22)

Case 4.

\[
k = \frac{\omega}{2 \mu}, \quad a_1 = -\frac{\tilde{a}\sqrt{6} \sqrt{\mu}}{\beta}, \quad a_0 = b_1 = 0
\]

(23)

where \( \omega \) is an arbitrary constant.
If $\frac{\omega}{\mu} < 0$, then $k < 0$, substituting (23) into (12) and by the use of (9), the solution of mKdV is given by:

$$u_{\gamma}(x,t) = \pm \frac{\tilde{a} \sqrt{3} \sqrt{\mu} \sqrt{-\frac{\omega}{\mu}}}{\sqrt{\beta}} \text{Tanh}\left[\sqrt{-\frac{\omega}{\mu}}(x-t\omega)\right]$$

(24)

If $\frac{\omega}{\mu} > 0$, then $k > 0$, substituting (23) into (12) and by the use of (9), the solution of mKdV is given by:

$$u_{\delta}(x,t) = -\frac{\tilde{a} \sqrt{3} \sqrt{\mu} \sqrt{\frac{\omega}{\mu}}}{\sqrt{\beta}} \text{Tanh}\left[\sqrt{-\frac{\omega}{\mu}}(x-t\omega)\right]$$

(25)

Fig. 1. Plot of solitary wave solution of (19).

2.2 The Modified Korteweg –de Vries Burgers (mKdVB) equation

According to the used method the mKdVB equation (3) becomes the ODE:

$$-\omega U' + \beta U'^2 + \nu U'' + \mu U''' = 0$$

(26)

Integrating (26) with zero constant of integration to have become:

$$-\omega U + \frac{\beta}{3} U^3 + \nu U' + \mu U'' = 0$$

(27)

with $m = 1$ (by balancing $U''$ with $U^3$), the finite expansion (8) would have the form:
\[ U(\xi) = a_0 + a_i \phi(\xi) + b_i \phi^{-1}(\xi) \]  

(28)

Substituting (28) into (27) and with the use of (7), the following algebraic system is obtained for the coefficients of \( \phi^i, i = 0, \pm 1, \pm 2, \pm 3, \)

\[
2k^2 \mu b_1 + \frac{\beta b_3}{3} = 0
\]
\[
-k v b_1 + \beta a_i b_i^2 = 0
\]
\[
2k \mu b_1 - \omega b_1 + \beta a_i^2 b_i + \beta a_i b_i^2 = 0
\]
\[
-\omega a_0 + \frac{\beta a_0^3}{3} + k v a_i - v b_1 + 2 \beta a_i b_i = 0
\]
\[
2k \mu a_i - \omega a_i + \beta a_i^2 a_i + \beta a_i^2 b_i = 0
\]
\[
v a_i + \beta a_i a_i^2 = 0
\]
\[
2 \mu a_i + \frac{\beta a_i^3}{3} = 0
\]

(29)

The system in (29) is solved to give the following sets of solutions, and by the sign test of \( k \) the exact solutions of modified KdV–Burgers equation (3) are obtained as follows:

**Case 1.**

\[ k = -\frac{9 \omega^2}{16 v^2}, a_0 = \mp \frac{1}{2} \sqrt{\frac{3 \omega}{2 \beta}}, b_1 = \pm \frac{3 \sqrt{3 \omega \beta^3}}{8 \sqrt{v^2}}, a_i = 0, \mu = -\frac{2 v^2}{9 \omega} \]  

(30)

where \( \omega \) is an arbitrary constant. As it is noted \( k \) would have a negative value (\( k < 0 \)) whenever \( \frac{\omega^2}{v^2} > 0 \) for a real \( \omega \) and \( v \). Returning the values of (30) into (28) and by the use of (9), the solution of modified KdV-Burgers (3) is:

\[
u(t,x) = \mp \frac{\sqrt{3 \omega}}{2 \sqrt{\beta}} \frac{9 \sqrt{5 \omega^3 \beta^3}}{32 \sqrt{5 \beta v}} \frac{\omega^2}{v^2} \text{Coth}\left[ \frac{3}{4} \left( \frac{\omega^2}{v^2} (x - t \omega) \right) \right] \]  

(31)

**Case 2.**

\[ k = -\frac{9 \omega^2}{16 v^2}, a_0 = \mp \frac{\sqrt{3 \omega}}{2 \sqrt{\beta}}, a_i = \pm \frac{2 v}{\sqrt{3 \beta \omega}}, b_1 = 0, \mu = -\frac{2 v^2}{9 \omega} \]  

(32)

where \( \omega \) is an arbitrary constant. As it is noted \( k \) would have a negative value (\( k < 0 \)) whenever \( \frac{\omega^2}{v^2} > 0 \) for a real \( \omega \) and \( v \). Returning the values of (32) into (28) and by the use of (9), the solution of modified KdV-Burgers (3) is:
\[ u_3(x,t) = -\frac{\sqrt{3} \sqrt{\omega}}{2 \sqrt{\beta}} - \frac{\sqrt{5} \nu}{2 \sqrt{\beta \omega}} \sqrt{\omega^2} \left\{ \frac{3}{4} \sqrt{\frac{\omega^2}{\nu^2}} (x-t\omega) \right\} \]  

(33)

Case 3.

\[ k = -\frac{9 \omega^2}{64 \nu^2}, \quad a_0 = \pm \frac{\sqrt{3} \sqrt{\omega}}{2 \sqrt{\beta}}, \quad a_i = \pm \frac{2 \nu}{\sqrt{3} \sqrt{\beta \omega}}, \quad b_i = \pm \frac{3 \sqrt{3} \omega^{3/2}}{32 \sqrt{\beta \nu}}, \quad \mu = -\frac{2 \nu^2}{9 \omega} \]  

(34)

where \( \omega \) is an arbitrary constant. As it is noted \( k \) would have a negative value (\( k < 0 \)) whenever \( \frac{\omega^2}{\nu^2} > 0 \), for a real \( \omega \) and \( \nu \). Returning the values of (34) into (28) and by the use of (9), the solution of modified KdV-Burgers (3) is:

\[ u_3(x,t) = -\frac{\sqrt{3} \sqrt{\omega}}{2 \sqrt{\beta}} + \frac{9 \sqrt{3} \omega^{3/2}}{256 \sqrt{\beta \nu}} \sqrt{\omega^2} \left\{ \frac{3}{8} \sqrt{\frac{\omega^2}{\nu^2}} (x-t\omega) \right\} \]

\[ \mp \frac{\sqrt{3} \sqrt{\omega}}{4 \sqrt{\beta \omega} \sqrt{\omega}} \left\{ \frac{3}{8} \sqrt{\frac{\omega^2}{\nu^2}} (x-t\omega) \right\} \]

(35)

Fig. 2. Plot of solitary wave solution of (35)

2.3 The Modified Kawahara (mK) equation

Following the methodology of the used method the modified Kawahara (mK) equation (4) becomes the ODE:
\[-\omega U' + \beta U^2 U' + \mu U''' - \delta U'''' = 0 \] (36)

Integrating (36) with zero constant of integration to become:
\[-\omega U + \frac{\beta}{3} U^3 + \mu U' - \delta U'' = 0 \] (37)

with \( m = 2 \) (by balancing \( U^{(4)} \) with \( U' \)), the finite expansion (8) would have the form:
\[ U(\xi) = a_0 + a_i \phi(\xi) + a_i \phi^2(\xi) + b_i \phi^{-1}(\xi) + b_i \phi^{-2}(\xi) \] (38)

Substituting (38) into (37) and with the use of (7), the following algebraic system is obtained for the coefficients of \( \phi^i, i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6 : \)
\[-120k^4 \delta b_2 + \frac{\beta b_3^2}{3} = 0 \]
\[-24k^4 \delta b_1 + \beta b_1 b_2^2 = 0 \]
\[-24k^4 \delta b_2 + 6k^2 \mu b_2 + \beta b_1^2 b_2 + \beta a_0 b_2^2 = 0 \]
\[-40k^3 \delta b_1 + 2k^2 \mu b_1 + \frac{\beta b_3^3}{3} + 2 \beta a_0 b_1 b_2 + \beta a_0 b_2^2 = 0 \]
\[\beta a_0 b_1^2 - 136k^2 \delta b_2 + 8k \mu b_2 - \omega b_2 + \beta a_0 b_2 + 2 \beta a_0 b_1 b_2 b_2 + \beta a_2 b_2^2 = 0 \]
\[-16k^2 \delta b_1 + 2k \mu b_1 - \omega b_1 + \beta a_0 b_1 + \beta a_1 b_1 + 2 \beta a_0 a_1 b_2 + 2 \beta a_2 b_1 b_2 = 0 \]
\[-\omega a_0 + \frac{\beta a_0^3}{3} - 16k^3 \delta a_2 + 2k^2 \mu a_2 + 2 \beta a_0 a_1 b_1 + \beta a_0 a_2 b_1 - 16k \delta b_2 + 2 \mu b_2 + \beta a_0^2 b_2 + 2 \beta a_0 a_2 b_2 = 0 \]
\[-16k^2 \delta a_1 + 2k \mu a_1 - \omega a_1 + \beta a_0 a_1 + \beta a_1 b_1 + 2 \beta a_0 a_2 b_1 + 2 \beta a_1 a_2 b_2 = 0 \]
\[\beta a_0 a_1^2 - 136k^2 \delta a_2 + 8k \mu a_2 - \omega a_2 + \beta a_0^2 a_2 + 2 \beta a_0 a_2 b_1 + \beta a_2 b_2^2 = 0 \]
\[-40k \delta a_1 + 2k \mu a_1 + \frac{\beta a_1^3}{3} + 2 \beta a_0 a_1 a_2 + \beta a_2 b_1 = 0 \]
\[-240k \delta a_2 + 6 \mu a_2 + \beta a_0^2 a_2 + \beta a_0 a_2^2 = 0 \]
\[-24 \delta a_1 + \beta a_1 a_2^2 = 0 \]
\[-120 \delta a_2 + \beta a_2^3 = 0 \] (39)

The system in (39) is solved to give the following sets of solutions, and by the sign test of \( k \) the exact solutions of \( mK \) equation are obtained as follows:

**Case 1.**
\[ k = -\frac{5 \omega}{22 \mu}, \quad a_2 = \pm \frac{3}{22} \sqrt{\frac{\mu}{2}} \sqrt{\frac{\mu}{2}}, \quad b_2 = \pm \frac{15}{22} \sqrt{\frac{5 \mu}{2}} \sqrt{\frac{\mu}{2}}, \quad a_0 = a_1 = b_1 = 0, \quad \delta = -\frac{11 \mu}{100 \omega} \] (40)

where \( \omega \) is an arbitrary constant.
If $\frac{\omega}{\mu} > 0$, then $k < 0$, substituting (40) into (38) and by the use of (9), the solution of mK is given by:

$$u_1(x,t) = \mp \frac{3}{22} \sqrt{\frac{-\mu^2}{\omega}} \frac{1}{\sqrt{\beta \mu}} \tanh \left[ \frac{5}{22} \sqrt{\frac{\omega}{\mu}} (x - tw) \right]^2$$

$$\pm \frac{75}{484} \sqrt{\frac{\mu^2}{\omega}} \frac{1}{\beta \mu^3} \coth \left[ \frac{5}{22} \sqrt{\frac{\omega}{\mu}} (x - tw) \right]^2$$

(41)

If $\frac{\omega}{\mu} < 0$, then $k > 0$, substituting (40) into (38) and by the use of (9), the solution of mK is given by:

$$u_2(x,t) = \mp \frac{3}{22} \sqrt{\frac{-\mu^2}{\omega}} \frac{1}{\sqrt{\beta \mu}} \tan \left[ \frac{5}{22} \sqrt{\frac{\omega}{\mu}} (x - tw) \right]^2$$

$$\pm \frac{75}{484} \sqrt{\frac{\mu^2}{\omega}} \frac{1}{\beta \mu^3} \coth \left[ \frac{5}{22} \sqrt{\frac{\omega}{\mu}} (x - tw) \right]^2$$

(42)

Case 2.

$$k = -\frac{5\omega}{16\mu}, a_0 = \mp \frac{3\delta}{2\sqrt{\beta}}, b_2 = \pm \frac{15\delta}{32\sqrt{\beta \mu}}, a_1 = a_2 = b_1 = 0, \delta = \frac{4\mu^2}{25\omega}$$

(43)

where $\omega$ is an arbitrary constant.

If $\frac{\omega}{\mu} > 0$, then $k < 0$, substituting (43) into (38) and by the use of (9), the solution of mK is given by:

$$u_3(x,t) = \mp \frac{3\delta}{2\sqrt{\beta}} \frac{5\sqrt{-\omega}}{2\sqrt{\beta \mu}} \pm \frac{75\delta}{512\sqrt{\beta \mu^2}} \coth \left[ \frac{1}{4} \sqrt{\frac{5}{\mu}} (x - tw) \right]^2$$

(44)

If $\frac{\omega}{\mu} < 0$, then $k > 0$, substituting (44) into (38) and by the use of (9), the solution of mK is given by:

$$u_4(x,t) = \mp \frac{3\delta}{2\sqrt{\beta}} \frac{5\sqrt{-\omega}}{2\sqrt{\beta \mu}} \pm \frac{75\delta}{512\sqrt{\beta \mu^2}} \coth \left[ \frac{1}{4} \sqrt{\frac{5}{\mu}} (x - tw) \right]^2$$

(45)

Case 3.

$$k = -\frac{5\omega}{16\mu}, a_0 = \mp \frac{3\delta}{2\sqrt{\beta}}, b_2 = \pm \frac{12\delta}{5\mu \sqrt{-\omega}} \sqrt{\beta \omega}, a_1 = b_1 = b_2 = 0, \delta = \frac{4\mu^2}{25\omega}$$

(46)

where $\omega$ is an arbitrary constant.
If $\frac{\omega}{\mu} > 0$, then $k < 0$, substituting (46) into (38) and by the use of (9), the solution of mK is given by:

$$u_5(x,t) = \mp \frac{3\sqrt{5}}{2} \sqrt{-\omega} + \frac{3\sqrt{5}}{2} \sqrt{-\omega} \pm \frac{3\sqrt{5}}{2} \sqrt{-\omega} \pm \frac{1}{4} \sqrt{5} \sqrt{\omega \mu} (x-t\omega)^2 \tag{47}$$

If $\frac{\omega}{\mu} < 0$, then $k > 0$, substituting (44) into (38) and by the use of (9), the solution of mK is given by:

$$u_6(x,t) = \mp \frac{3\sqrt{5}}{2} \sqrt{-\omega} - \frac{3\sqrt{5}}{2} \sqrt{-\omega} \pm \frac{3\sqrt{5}}{2} \sqrt{-\omega} \pm \frac{1}{4} \sqrt{5} \sqrt{-\omega \mu} (x-t\omega)^2 \tag{48}$$

**Case 4.**

$$k = -\frac{5\omega}{64\mu}, \ a_0 = \mp \frac{3\sqrt{5}}{4} \sqrt{\omega}, \ a_2 = \pm \frac{12\sqrt{5}}{8\sqrt{\mu}}, \ b_2 = \pm \frac{15\sqrt{5}}{512\sqrt{\mu}}, \ a_1 = b_1 = 0, \ \delta = \frac{4\mu^2}{25\omega} \tag{49}$$

where $\omega$ is an arbitrary constant.

If $\frac{\omega}{\mu} > 0$, then $k < 0$, substituting (49) into (38) and by the use of (9), the solution of mK is given by:

$$u_7(x,t) = \mp \frac{3\sqrt{5}}{4} \sqrt{\omega} \pm \frac{3\sqrt{5}}{4} \sqrt{\omega} \pm \frac{3\sqrt{5}}{4} \sqrt{\omega} \pm \frac{1}{8} \sqrt{5} \sqrt{\omega \mu} (x-t\omega)^2 \tag{50}$$

$$\pm \frac{75}{2} \frac{5^{5/2}}{32768\sqrt{\beta \mu^2}} \text{Cot}[\frac{1}{8} \sqrt{5} \sqrt{\omega \mu} (x-t\omega)^2]$$

If $\frac{\omega}{\mu} < 0$, then $k > 0$, substituting (49) into (38) and by the use of (9), the solution of mK is given by:

$$u_8(x,t) = \mp \frac{3\sqrt{5}}{4} \sqrt{\omega} \pm \frac{3\sqrt{5}}{4} \sqrt{\omega} \pm \frac{3\sqrt{5}}{4} \sqrt{\omega} \pm \frac{1}{8} \sqrt{5} \sqrt{-\omega \mu} (x-t\omega)^2 \tag{51}$$

$$\pm \frac{75}{2} \frac{5^{5/2}}{32768\sqrt{\beta \mu^2}} \text{Cot}[\frac{1}{8} \sqrt{5} \sqrt{-\omega \mu} (x-t\omega)^2]$$
3 Conclusion

In this study, we have successfully employed the modified extended Tanh function method with Riccati equation for obtaining the solitary travelling wave solutions for a given PDE's. The method has an advantage of being direct and concise. Enormous variety of solutions were obtained with the aid of Mathematica software.

Competing Interests

Authors have declared that no competing interests exist.

References


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