Positive Position Feedback Controller for Nonlinear Beam Subject to Harmonically Excitation

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this paper, the vibration reduction of the harmonically excited nonlinear beam is introduced using positive position feedback controller (PPF). The multiple-scale perturbation techniques (MSPT) up is applied to second-order to obtain the analytic results. Numerical simulations are used to compare between time-history and the analytical solution. The frequency response equation (FRE) is studied to illustrate the steady state solutions near the simultaneous resonances. The influences of the different parameters and the system behavior at resonance case are studied to show the optimum conditions of decreasing the vibration. A comparison between the numerical and analytical solutions is presented to appear the validity of the results.

Keywords: Nonlinear beam; multiple-scale method; nonlinear differential equations; resonance; PPF controller; stability.
1 Introduction

Vibration control is one of the most important research studies of dynamical systems. The theory and technique of vibration suppression have been extensively investigated for many years. In this respect, it was advocated in Ref. [1] that suppressing the serious vibrations is of high importance in the general engineering applications and in particular bifurcation control theory. Recently, many research papers have been devoted to study the control of resonantly forced systems. For instance, different ranking of vibration such as Free and forced vibration, undamped and damped vibration, linear and nonlinear vibration, and deterministic and unrestrained vibration were introduced [2]. In addition, the different procedures concerned in a vibration trial of an engineering system were discussed as well as the main definitions and connotations of vibration.

On the other hand, passive and active control methods have been applied to decrease the vibrations and dynamic chaos in both structures and machines [3]. Moreover, Optimum cooperative conditions of the main system had been extracted by applying both passive and active control methods. If the elasticity of the construction is large, nonlinear vibrations with big amplitudes will continually occur for a lengthy period under the external forces, which will inevitably affect the constructing normal work and even result in many disadvantages in the constructing tiredness. So, one needs to consider the active control of the constructing nonlinear vibration stability and response. Up to date, the problems of constructing nonlinear vibration and nonlinear dynamics have attracted many researchers’ interest. For example, the problems that focus on nonlinear dynamics modeling, nonlinear response computation, motion stability analysis, bifurcation and chaos characteristics, were discussed in [4-7].

The vibration stability and the active control of the parametrically exited nonlinear beam structures were studied [8], using the piezoelectric material. They applied the principle of Hamilton’s to obtain the nonlinear equation of motion. In that case, the multiple-scale technique was applied up to first-order to solve the cubic nonlinear equation of motion with damping. In Ref. [9], nonlinear control rule was proposed to suppress the vibrations of the first mode of a cantilever beam when subjected to a fundamental parametric excitation. Accordingly, it was emphasized that the major use of PPF control is that the frequency response of the controller rolls off quickly, making the locked loop system robust to spill over.

Moreover, the connection between the output feedback control and the PPF was investigated in Ref. [10]. Also, in Ref. [11] an active vibration control of a nonlinear beam with self and external excitations was studied by employing PPF controller. In addition, Ref. [12] presented a survey for PPF controller that was used to repress the vibration amplitude of a nonlinear dynamic model at primary resonance and the existence of 1:1 internal resonance. The optimal vibration control with model PPF controller was studied [13,14], where the belated feedback control and fullness control were considered in order to reduce the vibration of a dynamical system.

In Ref. [15] the torsional vibration of a nonlinear dynamical system can be reduced using the active negative velocity feedback. The MSPT approximation has been applied to find the solution for the ordinary differential equation that represented this system. Ref. [16] has derived the first order differential equations governing the time growth of the amplitudes and phases of both the system and the controller using MSPT. Then, the bifurcation analysis had been conducted to examine the stability of the locked loop system. Applying MSPT up to the second-order approximation [17], the solutions for the coupled differential equations can be obtained. In that case, the frequency response equations could be used to test the system’s stability close to the simultaneous primary and subharmonic resonances in the existence of internal resonance to the system with absorbers. The nonlinear system close to the simultaneous sub-harmonic in the presence of internal resonance, was studied [18]. The FRE are perturbed to research the stability of the steady-state solutions. Nonlinear vibration of Nanobeam based on thermal and surface effects were investigated in [19]. The Galerkin and the Multiple Scales techniques are applied to disband the problem. In addition, Ref. [20] investigated the primary resonance, dynamical stability, and bifurcations of the performance of a piecewise-smooth (PWS) system with negative hardness. The sensitivity of the controller parameters on the responses has been analyzed. The symmetry- fracturing bifurcation and the chaotic motion have been achieved.
The planning of the search is as follows. In Sect. 2, the mathematical model of a nonlinear Beam subject to harmonically excitation after adding PPF controller is introduced. Then, the technique of multiple scales is used to get an approximate solution for the system response. In Sect. 3, the stability condition of the nonlinear solution is specified. In Sect. 4, the numerical solution and the influences of some system parameters on the vibrating system are introduced and declared to displays the optimization conditions of controlling the vibration system. Finally, Sect. 5 shows the conclusion.

2 Mathematical Model

A diagrammatic sketch as presented in Fig. 1 displays the nonlinear beam subjected to axial forces. The harmonic force \( P(t) = F_0 + F \cos(\Omega t) \) is applied along the axial orientation of the base beam. The material of the base beam is homogeneous and isotropic. The dynamical behavior of the considered system, which is specified in Ref. [8], is described by the dimensionless nonlinear ordinary differential equation. Here this dynamical model will be modified by adding another equation to describe PPF controller as:

\[
\ddot{W} + 2 \xi_1 \omega_1 \dot{W} + \omega_1^2 W + \alpha W \ddot{W} + \beta W^3 + \gamma W^5 = F \cos(\Omega t) + q V
\]

\[
\dot{V} + 2 \xi_2 \omega_2 \dot{V} + \omega_2^2 V = p W
\]

Where \( W \) and \( V \) are displacement of the main system and controller, respectively, \( \omega_1 \) and \( \omega_2 \) are linear natural frequencies of main system and controller, respectively, \( \xi_1 \) and \( \xi_2 \) (\( \xi_{1,2} = \varepsilon \xi_{1,2} \)) are the damping coefficients of main system and controller are, respectively, \( \alpha, \beta \) and \( \gamma \) (\( \alpha = \varepsilon \alpha, \beta = \varepsilon \beta, \gamma = \varepsilon \gamma \)) are the nonlinear parameters of the main system, \( p \) is the Feedback signal gain (\( p = \varepsilon \hat{p} \)), \( q \) is the Control signal gain (\( q = \varepsilon \hat{q} \)) and \( \varepsilon \) is a small perturbation parameter (\( 0 < \varepsilon \leq 1 \)).

![Fig. 1. Schematic graph of beam undergone to axial forces](image)

2.1 Mathematical Analysis

To produce damping, nonlinearities, external excitation force amplitude, feedback gain, and control signal gain appear in the corresponding perturbation equations when applying MSPT, we scale the parameters of the equations as \( \xi_i = \varepsilon \xi_i (i = 1, 2), \alpha = \varepsilon \alpha, \beta = \varepsilon \beta, \gamma = \varepsilon \gamma, F = \varepsilon F, q = \varepsilon q, p = \varepsilon p \), where \( \varepsilon \) is a small dimensionless perturbation parameter and, \( 0 < \varepsilon \leq 1 \).

utilizing the multiple time scale perturbation method up to second approximation, solving Eqs. (1) and (2) analytically by seeking the solutions in the forms:

\[
W(t; \varepsilon) = \sum_{m=0}^{2} \varepsilon^m W_m(T_0, T_1, T_2) + O(\varepsilon^3)
\]
\[ V(t, \varepsilon) = \sum_{m=0}^{2} \varepsilon^m V_m(T_0, T_1, T_2) + O(\varepsilon^3) \]  
(4)

where \( T_0 = t \) is the fast-varying time scale, \( T_1 = \varepsilon t, T_2 = \varepsilon^2 t \) are the slowly varying time scales. Therefore, the time derivatives can be established as follows:

\[
\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 
\]  
(5)

\[
\frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2D_0 D_2) 
\]  
(6)

Where \( D_r = \frac{\partial}{\partial T_r}, r = 0, 1, 2 \).

Substituting Eqs. (3)-(6) into Eqs. (1), (2) and equating the coefficients of the same powers of \( \varepsilon \) result in the following:

Order \( (\varepsilon^0) \):

\[
(D_0^2 + \omega_1^2) W_0 = 0 
\]  
(7)

\[
(D_0^2 + \omega_2^2) V_0 = 0 
\]  
(8)

Order \( (\varepsilon^0) \):

\[
(D_0^2 + \omega_1^2) W_1 = -2D_0 D_1 W_0 - \alpha W_0 D_0 W_0 - 2\hat{\omega}_2 \omega_1 D_0 W_0 - \hat{\beta} W_0^2 - \hat{\gamma} W_0^3 + \hat{q} V_0 + \hat{F} \text{Cos}(\Omega t) 
\]  
(9)

\[
(D_0^2 + \omega_2^2) V_1 = -2D_0 D_1 V_0 - 2\hat{\omega}_2 \omega_2 D_0 V_0 + \hat{p} W_0 
\]  
(10)

Order \( (\varepsilon^2) \):

\[
(D_0^2 + \omega_1^2) W_2 = -D_0^2 W_0 - 2D_0 D_1 W_1 - 2D_0 D_2 W_0 - \alpha (W_0 (D_0 W_1 + D_1 W_0) + W_1 D_0 W_0) \\
-2\hat{\omega}_1 \omega_1 (D_0 W_1 + D_1 W_0) - 2\hat{\beta} W_0 W_1 - 3\hat{\gamma} W_0^2 W_1 + \hat{q} V_0 
\]  
(11)

\[
(D_0^2 + \omega_2^2) V_2 = -D_0^2 V_0 - 2D_0 D_1 V_1 - 2D_0 D_2 V_0 - 2\hat{\omega}_2 \omega_2 (D_0 V_1 + D_1 V_0) + \hat{p} W_1 
\]  
(12)

The general solutions of Eqs. of order \( (\varepsilon^0) \) represent the zero-order approximation and can be configured in the form:

\[ W_0 = A(T, T_1) \exp(i\omega_1 T_0) + \text{cc.} \]  
(13)


\[ V'_0 = B(T, T_1) \exp(i \omega T_0) + \text{cc.} \]  

(14)

where \( A, B \) are complex functions in \( T \) and \( T_1 \), \( \text{cc.} \) indicate the complex conjugate of the previous terms of equations (13)-(14).

Substituting equations (13) - (14) into Eqs. of order \( \varepsilon \) we get the following:

\[
\begin{align*}
(D_0^2 + \omega^2)W_1 &= q B \exp(i \omega T_0) - \left( \hat{\beta} A^2 + i \hat{\alpha} \omega, A^2 \right) \exp(2i \omega T_0) - \hat{\gamma} A^3 \exp(3i \omega T_0) - \hat{\beta} \bar{A} A \\
&\quad - \left( 3 \hat{\gamma} A^2 \bar{A} + 2 i \xi_1 \omega_1 A^2 + 2 i \omega_1 D_i A \right) \exp(i \omega T_0) + \frac{1}{2} \hat{F} \exp(i \Omega T_0) + \text{cc.} \\
(D_0^2 + \omega^2)W_1 &= -\left( 2 i \xi_2 \omega_2 B + 2 i \omega_2 D_i B \right) \exp(i \omega T_0) + \hat{\rho} A \exp(i \omega T_0) + \text{cc.} \\
(15)
\end{align*}
\]

For bounded solutions of equations (15) and (16), the coefficient of the secular terms, must be eliminated and the general solutions of these equations are acquired as follows:

\[
\begin{align*}
W_1 &= C_1(T_1, T_2) \exp(i \omega T_0) + C_2(T_1, T_2) \exp(2i \omega T_0) + C_3(T_1, T_2) \exp(3i \omega T_0) \\
&\quad + C_4(T_1, T_2) + C_5(T_1, T_2) \exp(i \Omega T_0) + \text{cc.} \\
V_1 &= C_6(T_1, T_2) \exp(i \omega T_0) + \text{cc.} \\
(17)
\end{align*}
\]

where \( C_j \) \((j = 1, 2, \ldots, 6)\) are complex functions in \( T \) and \( T_1 \), which are presented in the appendix.

Substituting Eqs. (13) - (14) and (17) - (18) into Eqs. of order \( \varepsilon^2 \), we find that:

\[
\begin{align*}
(D_0^2 + \omega^2)W_2 &= \left( -6 \hat{\gamma} A \bar{A} C_1 - 2 i C_i \xi_1 \omega \omega_2 - 2 i \omega_2 D_i C_i \right) \exp(i \omega T_0) + \left( -2 \hat{\beta} A \left( C_4 + \bar{C}_4 \right) \\
&\quad - 2 \hat{\beta} A C_2 + q \xi_1 C_3 - 3 \hat{\gamma} C_3 A^2 - D_i^2 A - i \hat{\alpha} \omega_1 A \left( C_4 + \bar{C}_4 \right) - i \hat{\alpha} \omega_1 \bar{A} C_2 \\
&\quad - 2 i \omega_1 D_i A - 2 i \xi_1 \omega_1 D_i A \right) \exp(i \omega T_0) + \left( -3 \hat{\gamma} A^2 \left( C_4 + \bar{C}_4 \right) - 6 \hat{\gamma} A \bar{A} C_2 \\
&\quad - 2 \hat{\beta} C_1 \bar{A} - 2 i \hat{\alpha} \omega_1 \bar{A} C_1 - 4 i \xi_1 \omega_1 C_2 - \hat{\alpha} A D_i A - 4 i \omega_1 D_i C_i \right) \exp\left( 2i \omega T_0 \right) \\
&\quad + \left( -6 \hat{\gamma} A \bar{A} C_1 - 2 \hat{\beta} C_1 A - 3 i \hat{\alpha} \bar{A} C_1 \omega_1 - 6 i \xi_1 \omega_1 C_2 - 6 i \omega_1 D_i C_i \right) \exp\left( 3i \omega T_0 \right) \\
&\quad + \left( -3 \hat{\gamma} A^2 C_2 - 2 \hat{\beta} C_2 A - 4 i \hat{\alpha} \omega_1 A C_1 \right) \exp\left( 4i \omega T_0 \right) + \left( -3 \hat{\gamma} A^2 C_3 \right) \exp\left( 5i \omega T_0 \right) \\
&\quad + \left( -2 \hat{\beta} A C_1 - i \hat{\alpha} A C_1 \left( \omega_2 + \omega_1 \right) \right) \exp\left( i \left( \omega_2 + \omega_1 \right) T_0 \right) + \left( -2 \hat{\beta} \bar{A} C_1 \right. \\
&\quad - i \hat{\alpha} A \left( \omega_2 - \omega_1 \right) \right) \exp\left( i \left( \omega_2 - \omega_1 \right) T_0 \right) + \left( -3 \hat{\gamma} A^2 C_1 \right) \exp\left( i \left( \omega_2 + 2 \omega_1 \right) T_0 \right) \\
&\quad + \left( -3 \hat{\gamma} A^3 C_1 \right) \exp\left( i \left( \omega_2 - 2 \omega_1 \right) T_0 \right) + \left( -6 \hat{\gamma} A \bar{A} C_1 - 2 i \Omega \xi_1 \omega_1 C_2 - 2 i \omega_1 D_i C_i \right) \times \exp\left( i \Omega T_0 \right) + \left( -2 \hat{\beta} A C_1 - i \hat{\alpha} A C_1 \left( \Omega + \omega_1 \right) \right) \exp\left( i \left( \Omega + \omega_1 \right) T_0 \right) \\
&\quad + \left( -2 \hat{\beta} \bar{A} C_1 - i \hat{\alpha} \bar{A} C_1 \left( \Omega - \omega_1 \right) \right) \exp\left( i \left( \Omega - \omega_1 \right) T_0 \right) + \left( -3 \hat{\gamma} A^2 C_1 \right) \times \exp\left( i \left( \Omega + 2 \omega_1 \right) T_0 \right) - \left( 3 \hat{\gamma} A \bar{A} C_1 \right) \exp\left( i \left( \Omega - 2 \omega_1 \right) T_0 \right) \\
&\quad + \left( -6 \hat{\gamma} A \bar{A} C_1 - 3 \hat{\gamma} \bar{A} \bar{A} \bar{A} - \hat{\alpha} \bar{A} D_i A \right) + \text{cc.} \\
(19)
\end{align*}
\]
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\[
(D_0^2 + \omega_0^2) V_2 = \left( \dot{p} C_1 - 2i \omega_2 D_2 B - 2\xi_2 \omega_2 D_2 C_0 - D_2^2 B \right) \exp(i \omega_2 T_0) + \left( \dot{p} C_2 \right) \exp(2i \omega_1 T_0) \\
\quad + \dot{p} C_3 + \left( -2i \xi_2 \omega_2 \omega_1 C_0 - 2i \omega_1 D_2 C_0 \right) \exp(i \omega_1 T_0) + \left( \dot{p} C_3 \right) \exp(3i \omega_1 T_0) \\
\quad + \dot{p} C_3 \exp(i \Omega T_0) + cc. 
\]  

(20)

Exclusion the secular terms from equations (19), (20), the general solutions of these equations are obtained as follows:

\[
W_2 = E_1(T_1, T_2) \exp(i \omega_2 T_0) + E_2(T_1, T_2) \exp(2i \omega_1 T_0) + E_3(T_1, T_2) \exp(3i \omega_1 T_0) + E_4 \exp(4i \omega_1 T_0) \\
\quad + E_5(T_1, T_2) \exp(5i \omega_1 T_0) + E_6(T_1, T_2) \exp(i(\omega_2 + \omega_1) T_0) + E_7(T_1, T_2) \exp(i \omega_2 T_0) + E_8(T_1, T_2) \exp(i \omega_1 T_0) \\
\quad + E_9(T_1, T_2) \exp(i \omega_2 - 2 \omega_1 T_0) + E_{10}(T_1, T_2) \exp(i \Omega T_0) \\
\quad + E_{11}(T_1, T_2) \exp(i(\Omega + \omega_1) T_0) + E_{12}(T_1, T_2) \exp(i(\Omega - \omega_1) T_0) + E_{13}(T_1, T_2) \exp(i(\Omega + 2 \omega_1) T_0) \\
\quad + E_{14}(T_1, T_2) \exp(i(\Omega - 2 \omega_1) T_0) + E_{15}(T_1, T_2) + cc. 
\]  

(21)

\[
V_2 = E_{16}(T_1, T_2) \exp(i \omega_1 T_0) + E_{17}(T_1, T_2) \exp(2i \omega_1 T_0) + E_{18}(T_1, T_2) \exp(3i \omega_1 T_0) \\
\quad + E_{19}(T_1, T_2) \exp(i \Omega T_0) + E_{20}(T_1, T_2) + cc. 
\]  

(22)

where \( E_v (v = 1, 2, ..., 20) \) are complex functions in \( T_1 \) and \( T_2 \), which are presented in the appendix. Then we can find the analytical solution of Eqs. (1), (2) by substituting Eqs. (13), (14), (17), (18), (21) and (22) into Eqs. (3), (4).

### 3 Periodic Solution

The steady state solutions close to the simultaneous (primary and internal) resonance case (\( \Omega \equiv \omega_1, \omega_2 \equiv \omega_1 \)) are investigated from the first-order approximation solution. So, we will insert the detuning parameters \( \sigma_1 \) and \( \sigma_2 \) such that:

\[
\Omega = \omega_1 + \sigma_1 = \omega_1 + \epsilon \sigma_1 \quad \text{and} \quad \omega_2 = \omega_1 + \sigma_2 = \omega_1 + \epsilon \sigma_2
\]  

(23)

Substituting equation (23) into equations (15) and (16), the solvability conditions of the first-order approximation are obtained from the eliminated secular terms and the following differential equations are obtained

\[
D_1 A = i \hat{\Gamma}_1 A \hat{A} + \hat{\Gamma}_2 A + i \hat{\Gamma}_3 \exp(i \hat{\sigma}_1 T_1) + i \hat{\Gamma}_4 \exp(i \hat{\sigma}_2 T_1)
\]  

(24)

\[
D_1 B = \hat{\eta}_1 B + i \hat{\eta}_2 A \exp(-i \hat{\sigma}_2 T_1)
\]  

(25)

where \( \Gamma_n = \epsilon \hat{\Gamma}_n (n = 1, 2, 3, 4) \), and \( \eta_r = \epsilon \hat{\eta}_r (r = 1, 2) \) are constants (see appendix)

From Eq. (5), can be expressed the derivative of \( A(T_1, T_2) \) and \( B(T_1, T_2) \) at the first-order with respect to \( t \) as:

\[
\frac{d}{dt} A = \epsilon D_1 A
\]  

(26)
To analyze the solution of equations (24)-(25), it is convenient to express $A(T_1, T_2)$, $B(T_1, T_2)$ in the polar form as:

$$A = \left(\frac{a_1}{2}\right) \exp(i\delta_1)$$

$$B = \left(\frac{a_2}{2}\right) \exp(i\delta_2)$$

where the steady-state amplitudes are $a_1$ and $a_2$, and the phases of the polar solutions of the main system and controller are $\delta_1$ and $\delta_2$, respectively.

Substituting Eqs. (24) - (25) and (28), (29) into Eqs. (26), (27) and equating the real and imaginary parts we get the next equations characterizing the modification of the amplitudes and phases of the response:

$$\dot{a}_1 = \Gamma_2 a_1 - 2\Gamma_3 \sin(\theta_1) - \Gamma_4 a_2 \sin(\theta_2)$$

$$\dot{\theta}_1 = \sigma_1 - \frac{\Gamma_1 a_1^2}{4} - \frac{2\Gamma_3}{a_1} \cos(\theta_1) - \frac{\Gamma_4 a_2}{a_1} \cos(\theta_2)$$

$$\dot{a}_2 = \eta_1 a_2 + \eta_2 a_1 \sin(\theta_2)$$

$$\dot{\theta}_2 = \sigma_2 - \frac{\Gamma_1 a_1^2}{4} - \frac{2\Gamma_3}{a_1} \cos(\theta_1) - \frac{\Gamma_4 a_2}{a_1} \cos(\theta_2) + \frac{\eta_2 a_1}{a_2} \cos(\theta_2)$$

where $\theta_1 = \hat{\sigma}_1 T_1 - \hat{\delta}_1 = \sigma_1 t - \delta_1$, $\theta_2 = \hat{\sigma}_2 T_1 - \hat{\delta}_2 = \sigma_2 t - \delta_1 + \delta_2$. Eqs. (30) - (33) are called the autonomous amplitude-phase modulating equations.

### 3.1 Equilibrium solution

At steady-state motion we have

$$\dot{a}_1 = \dot{a}_2 = \dot{\theta}_1 = \dot{\theta}_2 = 0$$

Substituting Eq. (34) into Eqs. (30) - (33), we get

$$\Gamma_2 a_1 = 2\Gamma_3 \sin(\theta_1) + \Gamma_4 a_2 \sin(\theta_2)$$

$$\sigma_1 - \frac{\Gamma_1 a_1^3}{4} = 2\Gamma_3 \cos(\theta_1) + \Gamma_4 a_2 \cos(\theta_2)$$
\[ \eta_1 a_2 = -\eta_2 a_1 \sin(\theta_2) \quad (37) \]

\[ (\sigma_1 - \sigma_2) a_2 = \eta_2 a_1 \cos(\theta_2) \quad (38) \]

By using Eqs. (35) - (38), we obtain
\[ a_2^2 = \frac{\eta_2^2}{\left(\eta_1^2 + (\sigma_1 - \sigma_2)^2\right)} a_1^2 \quad (39) \]
\[ (\Gamma_2 a_1 + Z_1 a_1)^2 + (a_1 (\sigma_1 - Z_3) + a_1^3 Z_2)^2 = 4 \Gamma_3^2 \quad (40) \]

Where \( Z_1, Z_2 \) and \( Z_3 \) are presented in the appendix.

Equations (39) and (40) are the FRE that utilized to characterize the system steady state solutions conductance for the practical state i.e. \((a_1 \neq 0, a_2 \neq 0)\).

### 3.2 Stability analysis

To explain the stability of the nonlinear solution of the obtained fixed points, assume that:
\[ a_n = a_{n1} + a_{n0}, \quad \theta_n = \theta_{n1} + \theta_{n0} \quad (41) \]

where \( a_{n0} \) and \( \theta_{n0} \) are the solutions of Eqs. (30)-(33) and \( a_{n1}, \theta_{n1} \) are perturbations which are supposed to be small compared with \( a_{n0} \) and \( \theta_{n0} \).

Substituting Eq. (41) into Eqs. (30)-(33), and keeping only the linear terms in \( a_{n1} \) and \( \theta_{n1} \), we acquire the next equations that can be established in the matrix form as:
\[
\begin{bmatrix}
\dot{a}_{n1} \\
\dot{\theta}_{n1} \\
\dot{a}_{n2} \\
\dot{\theta}_{n2}
\end{bmatrix} =
\begin{bmatrix}
r_{11} & r_{21} & r_{31} & r_{41} \\
\rho_{11} & r_{22} & r_{32} & r_{42} \\
\rho_{12} & r_{23} & r_{33} & r_{43} \\
\rho_{13} & r_{24} & r_{34} & r_{44}
\end{bmatrix}
\begin{bmatrix}
a_{n1} \\
\theta_{n1} \\
a_{n2} \\
\theta_{n2}
\end{bmatrix}
\]

where the above square matrix is the Jacobian matrix, \( r_{hh}, k = 1,2,3,4 \) and \( h = 1,2,3,4 \) are given in the Appendix.. The eigenvalues of the above system of equations are given as:
\[ \lambda^4 + r_1 \lambda^3 + r_2 \lambda^2 + r_3 \lambda + r_4 = 0 \quad (43) \]

where \( \lambda \) denotes eigenvalues of a Jacobian matrix, \( r_1, r_2, r_3 \) and \( r_4 \) are coefficients of Eq. (43). Routh–Hurwitz gauge is employed to set up the stability of the equilibrium solutions. If the real part of the eigenvalues is negative, then the periodic solution is stable; otherwise, it is unstable. According to the Routh-Hurwitz criterion, the necessary and sufficient conditions for all the roots of equation (43) to possess negative real parts are
\[ r_1 > 0, \quad r_1 r_2 - r_3 > 0, \quad r_3 \left( r_1 r_2 - r_3 \right) - r_1^2 r_4 > 0, \quad r_4 > 0 \quad (44) \]
4 Numerical Results

The Runge-Kutta fourth order method employing MATLAB 7.7 (R2008b) package (ode45) has been applied to define the numerical solution of the system (1)-(2) as obvious in Figs. (2, 3) at the selected value $F = 9, \alpha_1 = 3, \xi_1 = 0.03, \alpha = 0.01, \beta = 0.5, \gamma = 0.6, \Omega = \omega_1, \omega_2 = \omega_3, \xi_2 = 0.003, p = 5, q = 10$. Fig. 2 illustrates the response and the phase-plane for the system without any controller at the primary resonance case. Fig. 3 shows the response and the phase-plane for the system and the controller with PPF controller at the simultaneous resonance case. From numerical solution using Runge-Kutta method, the steady-state amplitude of the main system with PPF control was reduced by about 99.98% from its value without control. Also, if we compare Fig. 2 with Fig. 3, we can see a good vibration reduction and the effect of self-excitation is eliminated.

This means that the effectiveness of the absorber $E_p$, ($E_p$ = steady state amplitude of the main system without absorber / steady state amplitude of the main system with absorber) is about 6715.

4.1 Effect of different parameters on the system (sensitivity analysis of the system)

The Frequency response curve (FRC) of the beam without control under various values of the excitation force $F$ is displayed at Fig. 4. In the following obtained Figs, solid lines illustrate stable solutions while dashed lines identify unstable solutions. It can be noticed that there are unstable regions in the FRC curve, which correspond to a quasiperiodic motion. In these unstable regions, roots of the characteristic equation are conjugate complex numbers with positive real parts. In stable regions, the beam vibrates periodically. It can be seen that the steady-state displacement amplitude of the beam increases as the external excitation force $F$ increases. Also, the curve is bent to the right denoting a softening effect and the jump phenomenon appears clearly due to the domination of the nonlinearity.

Fig. 5 shows the phase plane portrait of the uncontrolled beam at $\sigma_1 = 0.08, F = 9$, and we found that If the eigenvalues take the complex form $(a \pm ib, a < 0, b > 0)$, the equilibrium point is classified as the asymptotically stable spiral (spirals in) point at $(2.83272, 0.170792)$ as in Ref [21], which appear in this figure.

The FRC given by Eqs. (33), (34) are solved numerically for the amplitudes $a_1$ and $a_2$ against the detuning parameter $\sigma_1$ as shown in Figs. 6, 8. Fig. 6(a) and Fig. 6(b) show the effects of the detuning parameter $\sigma_1$ on the steady-state amplitudes $a_1$ and $a_2$ of the main system and controller, respectively. Fig. 7 presents a comparison between FRC of an uncontrolled beam and a controlled beam. We get a good vibration suppression bandwidth as indicated this figure.

![Fig. 2. Response of the system without absorber at primary resonance case $\Omega = \omega_1$.](image-url)
The effects of the different parameters are investigated as presented in Fig. 8. Fig. 8(a) displays that the steady-state amplitude $a_1$ is a monotonic increasing function of increasing the external excitation force $F$; also Fig. 8(b) displays that the steady-state amplitude $a_2$ is a monotonic increasing function of increasing the external excitation force $F$. For increasing values of the nonlinear parameter $\gamma$ the steady state amplitude $a_1$ is monotonically decreasing with shifting the curve to the right side as shown in Fig. 8(c); similarly, Fig. 8(d) show that the steady-state amplitude $a_2$ is monotonically decreasing with shifting the curve in the right side function to increase the nonlinear parameter $\gamma$. Moreover, we found no change in the FRC when the nonlinear parameters $\alpha$ and $\beta$ values changed. From Fig. 8(e) we observed that accordingly, the values of the Linear natural Frequency of main system $\omega_1$ increases, the curve is narrowed with monotonic decreasing the peak steady state amplitude $a_1$ of the system; also from Fig. 8(f) we observed that for increasing values of the Linear natural Frequency of the main system $\omega_1$, the curve is narrowed with monotonic decreasing the controller peak displacement amplitude $a_2$. Also, from Fig. 8(g) and Fig. 8(i) we observed that for increasing values of the Feedback signal gain $p$ and the control signal gain $q$, respectively that leads to monotonic increases of the steady state amplitudes $a_1$ with breadth the curve.

Furthermore, Fig. 8(h) shows that the steady-state amplitude $a_2$ is a monotonic increasing with breadth the curve at increasing the Feedback signal gain $p$. contrarily, Fig. 8(j) shows that the steady-state amplitude $a_2$ is a monotonic decreasing with narrowed the curve at increasing the control signal gain $q$.

In Figs. 8(a–j), it is clear that there is good vibration suppression around $\sigma_1 = 0$, however, for large values of $|\sigma_1|$ apart from zero, there are two peak displacement amplitudes of the beam. To overcome this problem, we can tune the controller natural frequency as studied in Figs. 8(k, l). The controller natural frequency can be tuned by changing the internal detuning parameter $\sigma_2$ depending on $\omega_2 = \omega_1 + \sigma_2$. Fig. 8 (k) clears
that minimum beam steady state displacement amplitude occurs when $\sigma_1 = \sigma_2$ i.e. $\Omega = \omega_2$. From this result, we can recommend tune the controller natural frequency to be equal to excitation frequency for dynamical systems, which are subjected to variable excitation frequency out the illustrated vibration suppression bandwidth in Fig.7. Also, in Fig. 8(k) the effect of the increase of on the amplitude $a_1$ leads to reach the amplitude at the zero in the different value $\sigma_2$ and made monotonic increasing and the curve is shifted to the right side; similarly in Fig. 8(l) and the effect of the increase of $\sigma_2$ on the amplitude $a_2$ made it monotonic increasing and the curve is shifted to the right side.

The effect of varying controller natural frequency $\omega_2$ on FRC of the beam and the controller is studied in Fig. 9 at different values of $\omega_1$. We change controller natural frequency $\omega_2$ by varying value of internal detuning parameter $\sigma_2$ because $\omega_2 = \omega_1 + \sigma_2$. The minimum steady-state displacement amplitude of the beam occurs at $\sigma_1 = \sigma_2$ which ensures the result of Fig.8(k, l). This is the optimal case for controller operation ($\sigma_1 = \sigma_2$).

Finally, Force-amplitude response curve for the beam and the controller before and after control is presented in Fig. 10 under the condition $\sigma_1 = \sigma_2$ for controlled beam. We notice that the connection between the beam displacement amplitude and the excitation force amplitude before using a controller is a nonlinear relation, which makes great beam displacement amplitudes for a small increase in the excitation force amplitude. After using the controller, the relation becomes linear in a good bandwidth of excitation force amplitude.

**Fig. 4.** The effect of the different values the external excitation force $F$ of the uncontrolled beam when $a_2 = 0$

**Fig. 5.** Phase plane of the uncontrolled beam when $F = 9$, $\sigma_1 = 0.08$, $a_2 = 0$
Fig. 6. The Frequency response curve of (a) the main system ($a_1$ against $\sigma_1$), and (b) the controller ($a_c$ against $\sigma_1$).

Fig. 7. Comparison between the FRC of an Uncontrolled beam and controlled beam.
Fig. 8. The effect of various parameters on the FRC

Fig. 9. The effect of different values of $\sigma_1$ on the FRC of (a) the main system ($a_1$ against $\sigma_1$), and (b) the controller ($a_2$ against $\sigma_2$)

Fig. 10. Beam and controlled force response curves for $\sigma_1 = \sigma_2 = 0$
4.2 Comparison between time response solutions of the Perturbation and the numerical methods

Fig. 11 declares the comparison between analytical solutions given by Eqs. (24) -(27) and the numerical solution of Eqs. (1)–(2) for PPF control when the condition for a steady-state solution is satisfied, that is, $\dot{a}_1 = \dot{a}_2 = \dot{\theta}_1 = \dot{\theta}_2 = 0$ at the primary resonance. The dashed lines appear the modulation of the amplitudes for the generalized coordinate $W, V$. However, the continuous lines appear the time history of vibrations, which acquired numerically as solutions of the original equations of the system with PPF controller. We realize that the asymptotic approximation specified by Eqs. (24) - (27) cannot create all the details of the dynamics of the original system given by Eqs. (1)- (2). For the sake of a good agreement between both analytical and numerical solutions, we may need a higher-order expansion ($O(\epsilon^3), O(\epsilon^4)$) and put it in the polar form with the resonance condition and it is a lengthy job in the present paper.

![Graph](image)

Fig. 11. (-----) analytic solution (-----) numerical solution with the same values of parameters at the primary resonance case $\Omega = \omega_1, \omega_2 = \omega_1$

4.3 Comparison between RK-4 and FRC.

Fig. 12 displays that the validated between the steady-state amplitude using the numerical solution (RK-4) of equations (1)-(2) (marked as small circles) and the steady-state amplitude using the FRE. Also, by dependence on Fig. 12, there is a good agreement with the amplitude using RK-4 at the similar value for the detuning parameter $\sigma_1$.

![Graph](image)

Fig. 12. Frequency response curves: (a) the main system ($a_1$ against $\sigma_1$), and (b) the controller ($a_2$ against $\sigma_1$)
5 Concluded Remarks

The vibration reduction of a nonlinear beam subjected to external excitation force has been presented by using PPF controller. MSPT up to the second approximation is investigated to define the FRE near the simultaneous (primary and internal) resonance. Employing the FRE the stability of the system has been displayed. A numerical solution of the system behavior without and with a controller is studied. From the above study, the following may be concluded:

1. Using positive position feedback, the steady-state amplitude is reduced to 99.98 % from its value without control.
2. The effectiveness of the controller $E_a$ is about 6715.
3. The steady-state amplitudes of the system and the PPF controller are a monotonic increasing function of increasing the excitation force $F$, the feedback signal gain $P$, the control signal gain $q$, and the detuning parameter $\sigma_2$.
4. The steady-state amplitudes of the system and the PPF controller are a monotonic decreasing function of increasing the nonlinear parameters $\gamma$ and the linear natural frequencies of the main system $\omega_1$.
5. For large values of $\sigma_1$ apart from zero, there are two peak displacement amplitudes of the beam. This problem can be solved by tuning the controller natural frequency such that $\Omega = \omega_2$ because the minimum beam steady state displacement amplitude occurs at $\sigma_1 = \sigma_2$. From this result, it is recommended to tune the controller natural frequency to be equal to excitation frequency for dynamical systems subjected to variable excitation frequency out the original vibration suppression bandwidth around $\sigma_1 = 0$. This tuning procedure can be applied practically if the average of change of excitation frequency can be accompanied by tuning controller natural Frequency ($\Omega = \omega_2$).
6. After using the controller, the relation between the beam displacement amplitude and the excitation force amplitude become linear in a good bandwidth of excitation force amplitude.
7. A comparison of the analytic results and the solutions obtained by Runge-Kutta fourth-order accuracy numerical method is illustrated which prove a good agreement.
8. A Comparison between RK-4 and FRC is presented which proves a good agreement.

Competing Interests

Authors have declared that no competing interests exist.

References


APPENDIX

\[
C_1 = \frac{\hat{q} B}{\omega^2_1 - \omega^2_2}, \quad C_2 = \frac{\hat{\beta} + i \hat{\alpha} \omega_1}{\frac{8}{3} \omega^2_1}, \quad C_3 = \frac{\hat{\beta} A \bar{A}}{\omega^2_1}, \quad C_4 = \frac{-\hat{\beta} A \bar{A}}{2(\omega^2_1 - \Omega^2)}, \quad C_5 = \frac{\hat{F}}{2},
\]

\[
C_6 = \frac{\hat{p} A}{\omega^2_2 - \omega^2_3}, \quad E_1 = \frac{\left(-6 \hat{\gamma} A \bar{A} C_1 - 2i C_1 \xi_1 \omega_1 \omega_2 - 2i \omega_2 D_i C_1 \right)}{\omega^3_1 - \omega^3_2},
\]

\[
E_2 = \frac{-1}{3 \omega_1^3} \left(-3 \hat{\gamma} A^2 C_4 + \bar{C}_4 - 6 \hat{\gamma} A \bar{A} C_2 - 2 \hat{\beta} C_3 \bar{A} - 2i \hat{\alpha} \omega_1 \bar{A} C_3
\]

\[
-4i \hat{\xi}_1 \omega_1^2 C_2 - \hat{\alpha} A D_i A - 4i \omega_1 D_i C_2 \right),
\]

\[
E_3 = \frac{\left(-6 \hat{\gamma} A \bar{A} C_3 - 2 \hat{\beta} C_2 A - 3i \hat{\alpha} AC_2 \omega_1 - 6i \hat{\xi}_1 \omega_2^2 C_3 - 6i \omega_1 D_i C_3 \right)}{-8 \omega_1^3},
\]

\[
E_4 = \frac{\left(-3 \hat{\gamma} A^2 C_2 - 2 \hat{\beta} C_3 A - 4i \hat{\alpha} \omega_1 AC_3 \right)}{-15 \omega_1^3}, \quad E_5 = \frac{3 \hat{\gamma} A^2 C_5}{24 \omega_1^3},
\]

\[
E_6 = \frac{\left(2 \hat{\beta} + i \hat{\alpha} \left(\omega_2 + \omega_1 \right) \right) AC_1}{\omega_1^3 + 2 \omega_1 \omega_2}, \quad E_7 = \frac{\left(2 \hat{\beta} + i \hat{\alpha} \left(\omega_2 - \omega_1 \right) \right) \bar{A} C_1}{\omega_1^3 - 2 \omega_1 \omega_2},
\]

\[
E_8 = \frac{3 \hat{\gamma} A^2 C_1}{3 \omega_1^3 + \omega_2^3 + 4 \omega_1 \omega_2}, \quad E_9 = \frac{3 \hat{\gamma} \bar{A}^2 C_1}{3 \omega_1^3 + \omega_2^3 - 4 \omega_1 \omega_2},
\]

\[
E_{10} = \frac{\left(6 \hat{\gamma} A \bar{A} C_5 + 2i \Omega \xi_1 \omega_1 C_5 + 2i \Omega D_i C_5 \right)}{\left(\Omega^2 - \omega_1^2 \right)},
\]

\[
E_{11} = \frac{\left(2 \hat{\beta} AC_5 + i \hat{\alpha} AC_5 \left(\Omega - \omega_1 \right) \right)}{\left(\Omega^2 + 2 \omega_1 \Omega \right)}, \quad E_{12} = \frac{\left(2 \hat{\beta} \bar{A} C_5 + i \hat{\alpha} \bar{A} C_5 \left(\Omega - \omega_1 \right) \right)}{\left(\Omega^2 - 2 \omega_1 \Omega \right)},
\]

\[
E_{13} = \frac{3 \hat{\gamma} A^2 C_5}{3 \omega_1^3 + \Omega^2 + 4 \Omega \omega_1}, \quad E_{14} = \frac{3 \hat{\gamma} \bar{A}^2 C_5}{3 \omega_1^3 + \Omega^2 - 4 \Omega \omega_1},
\]

\[
E_{15} = \frac{-1}{\omega_1^3} \left(6 \hat{\gamma} A \bar{A} C_4 + 3 \hat{\gamma} A \bar{A} C_2 + \hat{\alpha} \bar{A} D_i A \right), \quad E_{16} = \frac{1}{\omega_1^3 - \omega_2^3} \left(2i \hat{\xi}_1 \omega_1 \omega_2 C_6 + 2i \omega_1 D_i C_6 \right),
\]

\]

\]

\]

\]

\]

\]
\[ E_{17} = \frac{\hat{p}C_2}{\omega_1^2 - 4\omega_1^2}, \quad E_{18} = \frac{\hat{p}C_3}{\omega_1^2 - 9\omega_1^2}, \quad E_{19} = \frac{\hat{p}C_5}{\omega_2^2 - \Omega^2}, \quad E_{20} = \frac{\hat{p}C_4}{\omega_2^2}. \]

\[ \hat{F}_1 = \frac{3\hat{\eta}}{2\omega_1}, \quad \hat{F}_2 = -\hat{\eta}_{21}, \quad \hat{F}_3 = -\frac{\hat{\eta}_{22}}{4\omega_1}, \quad \hat{F}_4 = -\frac{\hat{\eta}_{23}}{2\omega_1}, \quad \hat{\eta}_1 = -\frac{\hat{\eta}_{24}}{\omega_2}, \quad \hat{\eta}_2 = -\frac{\hat{\eta}_{25}}{2\omega_2}. \]

\[ Z_1 = \frac{\Gamma_4 \eta_{11}}{(\eta_1^2 + (\sigma_1 - \sigma_2)^2)}, \quad Z_2 = \frac{\Gamma_1}{4}, \quad Z_3 = \frac{\Gamma_4 \eta_{12} (\sigma_1 - \sigma_2)}{(\eta_1^2 + (\sigma_1 - \sigma_2)^2)}. \]

\[ r_{11} = \Gamma_2, \quad r_{12} = -2\Gamma_3 \cos(\theta_{20}), \quad r_{13} = -\Gamma_4 \sin(\theta_{20}), \quad r_{14} = -\Gamma_4 a_2 \cos(\theta_{20}), \]

\[ r_{31} = \eta_2 \sin(\theta_{20}), \quad r_{32} = 0, \quad r_{33} = \eta_1, \quad r_{34} = \eta_2 a_1 \cos(\theta_{20}), \]

\[ r_{41} = \frac{\sigma_3}{a_{10}} - \frac{3}{4} \Gamma_1 a_{10} + \frac{2\eta_2}{a_{20}} \cos(\theta_{20}), \quad r_{42} = \frac{2\Gamma_3}{a_{10}} \sin(\theta_{10}), \]

\[ r_{43} = \frac{\sigma_3}{a_{20}} - \frac{a_{10}^2}{4a_{20}} \Gamma_1 - \frac{2\Gamma_3}{a_{20} a_{10}} \cos(\theta_{10}) - \frac{2\Gamma_4}{a_{10}} \cos(\theta_{20}), \]

\[ r_{44} = \left( \frac{a_{20} \Gamma_4}{a_{10}} - \frac{\eta_2}{a_{20}} \right) \sin(\theta_{20}). \]

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