Initial Value Problems for Second Order Neutral Impulsive Integro-differential Equations with Advanced Argument

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Abstract
This paper discusses initial value problems for second order neutral impulsive integro-differential equations with advanced argument. By using the fixed point theorem of either Leray-Schaude or Banach, two existence results are obtained. By comparison, each of them has his own strong and weak points. If appropriate changes are made to some conditions for two results, the same results can be got. Two examples to illustrate our main results are given, which are compared with the existence results for impulsive differential equations from existing literature.

Keywords: Neutral impulsive integro-differential equation; Second order; Initial value; Fixed point

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1 Introduction

Impulsive differential equations are now recognized as an excellent source of models to simulate processes and phenomena observed in control theory, physics, chemistry, population dynamics, biotechnology, industrial robotic, optimal control, etc. About initial value problems for impulsive differential equations, many authors have obtained very good existence results (for example, see [1-7]). Now consider the following equation

$$\begin{cases}
\{u(\phi(t))\}'' = f(t, u(t), u'(t), Ku(t), Hu(t)), & t \in J = [0, a], \ t \neq \xi_k, \\
\Delta u(t_k) = I_{0k}(u(t_k)), & \Delta u'(t_k) = I_{1k}(u'(t_k)), \ k = 1, \ldots, p,
\end{cases}$$  \tag{1.1}

where $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = a, \ \phi \in C^2(J, \mathbb{R}), \ \phi$ is monotone increasing with $t \leq \phi(t) \leq a (t \in J), \ \phi(0) = 0, \ \phi(a) = a, \ \phi'(t) > 0$ with $\phi^{-1} \in C^2(J, \mathbb{R}),$ and let $\phi(\xi_k) = t_k (k = 1, \ldots, p), \ J^* = J \setminus \{t_1, \ldots, t_p\}, \ J = J \setminus \{\xi_1, \ldots, \xi_p\}, \ f : J \times \mathbb{R} \to \mathbb{R}$ is continuous everywhere except at $\{\xi_k \times \mathbb{R}, f(\xi_k, x, x', y_1, y_2)\}$ and $f(\xi_k, x, x', y_1, y_2) = f(\xi_k, x, x', y_1, y_2),$ and $Ku(t) = \int_0^t h(t, s)u(s)ds, \ Hu(t) = \int_0^\infty h(t, s)u(s)ds, \ h(t, s) \in C(D \times \mathbb{R}^+),$ $\{t, s \in \mathbb{R}^2, 0 \leq s \leq t \leq a\}, \ k_0 = \max\{k(t, s) : (t, s) \in D\}, \ b_0 = \max\{h(t, s) : (t, s) \in J \times J\},$ further and $I_{0k}, I_{1k} \in C(\mathbb{R}, \mathbb{R}), \ \Delta u(t_k) = u(t_k) - u(t_k), \ \Delta u'(t_k) = u'(t_k) - u'(t_k).$ Denote by $PE(X, Y),$ where $X \subset Y \subset R,$ the set of all functions $u : X \to Y$ which are piecewise continuous in $X$ with points of discontinuity of the first kind at the points $t_k \in X,$ i.e., there exist the limits $u(t_k^-) < \infty$ and $u(t_k^+) = u(t_k) < \infty.$

2 Preliminaries

According to the properties of $\phi,$ there exist positive constants $m_1$ and $m_2$ such that $m_1 \leq \phi'(t) \leq m_2$ for all $t \in J.$

Let $E_0 = \{u|u, u' \in PE(J, \mathbb{R})\} \cap C^2(J, \mathbb{R}).$ Evidently, $E_0$ is a real Banach space with norm $\|u(t)\|_{E_0} = \max\{|u(t)|, |u'(t)|\}.$

Further, let $E = \{u|u(t) \in E_0\}.$ We can check that $E$ is also a real Banach space with norm $\|u(t)\|_{E_0} = \max\{|u(t)|, |u'(t)|\}.$

Define operator $B : u(t) \to u(\phi(t)),$ where $u(t) \in E_0$ and $u(\phi(t)) \in E.$ It is evident that $B$ is topological linear isomorphic, which implies that $E$ is a real Banach space.

Since $\frac{\phi(a) - \phi(0)}{a - 0} = \phi'(\bar{t}) (0 < \bar{t} < a),$ i.e., $\phi'(\bar{t}) = 1,$ we get $m_2 \geq 1,$ next $\|u(\phi(t))\|_{E_0} = m_2\|u'(t)\|_{E_0} \geq \|u'(t)\|_{E_0},$ so

$$\|u(t)\|_{E_0} \leq \|u(\phi(t))\|.  \tag{2.1}$$

Lemma 2.1. $u(t) \in E_0$ is a solution of (1.1) if and only if $u(t) \in E_0$ is a solution of the following integral equation

$$u(\phi(t)) = u_0 + u_0't + \int_0^t (t - s)f(s, u(s), u'(s), Ku(s), Hu(s))ds + \sum_{0 < \xi_k < t} [I_{0k}(u(t_k)) + (t - \xi_k)I_{1k}(u'(t_k))], \ t \in J.  \tag{2.2}$$

Proof. (i) Necessity

For $\xi_k < t \leq \xi_{k+1} (k = 0, 1, \ldots, p),$ by (1.1), we get

$$u(t) - u(0) = u(\phi(\xi_0)) - u(0) = \int_0^{\xi_1} (u(\phi(s)))'ds.$$
Leray-Schauder [6] Let the operator
Compactness criterion [7]

Similarly, we also have
\[
\int_{t_{k-1}}^{t_k} (u(\phi(s)))' \, ds,
\]

\[
\int_{t_{k-1}}^{t_k} (u(\phi(s)))' \, ds,
\]

\[
\int_{t_{k-1}}^{t_k} (u(\phi(s)))' \, ds.
\]

Substituting (2.4) into (2.3), it is easy to get (2.2).

Similarly, we obtain
\[
(u(\phi)(t))' = u_0' + \int_0^t (u(\phi(s)))' \, ds + \sum_{0 < \xi_k < t} I_{1k}(u'(t_k)), \; t \in J.
\]

Substituting (2.4) into (2.3), it is easy to get (2.2).

(ii) Sufficiency

According to (2.2), it is clear that
\[
u(0) = u_0, \quad \Delta u(t_k) = I_{0k}(u(t_k)).
\]

Differentiating both sides of (2.2), we have
\[
(u(\phi)(t))' = u_0' + \int_0^t f(s, u(s), u'(s), Ku(s), Hu(s)) \, ds + \sum_{0 < \xi_k < t} I_{1k}(u'(t_k)), \; t \in J.
\]

Similarly, we also have
\[
(u(\phi)(t))'' = f(t, u(t), u'(t), Ku(t), Hu(t)), \; t \in \bar{J}.
\]

By(2.6), it is evident that
\[
u'(0) = u_0', \quad \Delta u'(t_k) = I_{1k}(u'(t_k)).
\]

From (2.5),(2.7) and (2.8), we get that $u(t)$ is a solution of (1.1).

\[\Box\]

**Lemma 2.2.** (Leray-Schauder [6]) Let the operator $A : X \to X$ be completely continuous, where $X$ is a real Banach space. If the set $G = \{ \|x\| \leq x, x = \lambda Ax, 0 < \lambda < 1 \}$ is bounded, then the operator $A$ has at least one fixed point in the closed ball $T = \{ x \in X, \|x\| \leq R \}$, where $R = \sup G$.

**Lemma 2.3.** (Compactness criterion [7]) $H \subset PC(J, R)$ is a relatively compact set if and only if $H \subset PC(J, R)$ is uniformly bounded and equicontinuous on every $J_k$ $(k = 0, \cdots, p)$, where $J_0 = [a, t_1], J_k = (t_k, t_{k+1})$ $(k = 1, \cdots, p)$. 

\[\Box\]
3 Main Results

Let us introduce the following conditions for later use:

(H1) There exist nonnegative constants \(b, c, d, (i = 1, 2)\), \(b_k, c_k\) for \(k = 1, \cdots, p\), and \(g \in L(J, \mathbb{R}^+\) such that

\[
|f(t, x_1, y_1, z_1, z_2) - f(t, x_2, y_2, z_1, z_2)| \leq g(t)(b|x_2 - x_1| + c|y_2 - y_1| + \sum_{i=1}^p d_i|z_i - z_i|)
\]

\(t \in J\),

\[
|I_{ab}(x(t)) - I_{ab}(y(t))| \leq b_k|x_{k}(t) - y_{k}(t)|, \quad I_{ab}(0) = 0, \quad k = 1, \cdots, p,
\]

where \(x, y \in E_0\), \(y(t) = \phi(t), \) \(y(t) \in E_0\), \(z_{1i}, z_{2i}, \bar{z}_{1i}, \bar{z}_{2i}\)

(i = 1, 2) \(\in E_0\), \(a_0 = \int_0^t g(t)dt\).

(H2) There exist positive constant \(M\) such that \(|f(t, u(t), \phi(t), H_u(t))| \leq M(1 + \|u(t)\|_{E_0})\).

(H3) \(l = \max\{l_1, l_2\} < 1\), where \(l_1 = a^2M + \sum_{k=1}^p (b_k + a c_k)\), \(l_2 = \frac{m_2}{m_1}(a M + \sum c_k)\).

(H4) \(r = \max\{r_1, r_2\} < 1\), where \(r_1 = a a_0(b + c + a d_1 k_0 + a d_2 h_0) + \sum_{k=1}^p (b_k + a c_k)\),

\(r_2 = \frac{m_2}{m_1}[a_0(b + c + a d_1 k_0 + a d_2 h_0) + \sum_{k=1}^p c_k]\).

Theorem 3.1. If conditions (H1), (H2) and (H3) are satisfied, then (1.1) has at least one solution in the closed ball \(\overline{B} = \{u(\phi(t))|u(\phi(t)) \in E, \|u(\phi(t))\| \leq R\}, \) where \(R = \sup G, G = \{\|u(\phi(t))\|, u(\phi(t)) \in E, \|u(\phi(t))\| \leq \lambda A u(\phi(t))\}, \) \(0 < \lambda < 1\).

Proof. (i) For any \(u(\phi(t)) \in E\) define the operator \(A\) by

\[
Au(\phi(t)) = u_0 + u'_t + \int_0^t (t - s)f(s, u(s), u'(s), Ku(s), H_u(s))ds + \sum_{0<\xi_k<t} I_{ab}(u(t_k)) + (t - \xi_k)I_{ab}(u'(t_k)), \quad t \in J.
\]

It is easy to see that \(Au(\phi(t)) \in E_0\). According to the properties of \(\phi\), for any \(v(t) \in E_0\), we have \(v(t) = v(\phi^{-1}(\phi(t))) = v\phi^{-1}(\phi(t))\). Let \(u = v\phi^{-1}\). Next, it is clear that \(v(t) = u(\phi(t)) \in E\). It follows that \(A\) maps \(E\) to \(E\). Thus \(Au(\phi(t)) \in E\) with

\[
(Au(\phi(t)))' = u'_0 + \int_0^t f(s, u(s), u'(s), Ku(s), H_u(s))ds + \sum_{0<\xi_k<t} I_{ab}(u'(t_k)), \quad t \in J.
\]

A is a completely continuous operator will be verified by the following three steps.

Step 1. \(\phi\) is continuous.

Let any \(u_n(\phi(t)) \in \{u_n(\phi(t)) \in E, \|u_n(\phi(t)) - u(\phi(t))\| \to 0\} \) as \(n \to \infty\).

By (3.1) and (H1), we have

\[
|Au_n(\phi(t)) - Au(\phi(t))| \leq \int_0^t (t - s)g(s)
\]

\[
\sum_{0<\xi_k<t} b_k|u_n(t_k) - u(t_k)| + (t - \xi_k)c_k|u'_n(t_k) - u'(t_k)|\]

\[
\leq (b + c + a d_1 k_0 + a d_2 h_0)\|u_n(t) - u(t)\|_{E_0} \int_0^t (t - s)g(s)ds + \sum_{0<\xi_k<t} b_k + (t - \xi_k)c_k\]

\[
|Au_n(\phi(t)) - Au(\phi(t))| \leq \left[a_0(b + c + a d_1 k_0 + a d_2 h_0) + \sum_{k=1}^p (b_k + a c_k)\right]\|u_n(t) - u(t)\|_{E_0}, \quad t \in J.
\]

(3.3)
Then from (3.3) and (2.1), we have

$$
\|Au_n(\phi(t)) - Au(\phi(t))\|_{PC} \leq [a_0(b + c + ad_1k_0 + ad_2h_0) + \sum_{k=1}^{p}(b_k + ac_k)]|u_n(t) - u(t)|_{E_0},
$$

$$
\|Au_n(\phi(t)) - Au(\phi(t))\|_{PC} \leq [a_0(b + c + ad_1k_0 + ad_2h_0) + \sum_{k=1}^{p}(b_k + ac_k)]\|u_n(\phi(t)) - u(\phi(t))\|. \tag{3.4}
$$

Thus

$$
\|Au_n(\phi(t)) - Au(\phi(t))\|_{PC} \to 0 \text{ as } n \to \infty. \tag{3.5}
$$

Similarly, from (3.2) and (2.1), we get

$$
\left| \frac{d[Au_n(\phi(t)))] - Au(\phi(t))}{d\phi(t)} \right| \leq \frac{1}{m_1} \{a_0(b + c + ad_1k_0 + ad_2h_0) + \sum_{k=1}^{p}c_k\|u_n(\phi(t)) - u(\phi(t))\|, t \in J,
$$

$$
\|Au_n(\phi(t)) - Au(\phi(t))\|_{PC} \leq \frac{m_2}{m_1} \{a_0(b + c + ad_1k_0 + ad_2h_0) + \sum_{k=1}^{p}c_k\|u_n(\phi(t)) - u(\phi(t))\|. \tag{3.6}
$$

Thus

$$
\|Au_n(\phi(t)) - Au(\phi(t))\|_{PC} \to 0 \text{ as } n \to \infty. \tag{3.7}
$$

By (3.5) and (3.7), it is easy to see that \(\|Au_n(\phi(t)) - Au(\phi(t))\| \to 0 \text{ as } n \to \infty\), that is to say, \(A\) is continuous.

Step 2. \(A\) maps any bounded subset of \(E\) into one bounded subset of \(E\).

Let \(T\) be any bounded subset of \(E\). Then there exist \(h > 0\) such that \(\|u(\phi(t))\| \leq h\) for all \(u(\phi(t)) \in J\).

By (3.1), (H1), (H2) and (2.1), we have

$$
|Au(\phi(t))| \leq |u_0| + |u_0'|t + \int_0^t(t-s)M(1 + \|u(s)\|_{E_0})ds + \sum_{0 < \xi < t} [b_k|u(t_k)| + (t - \xi)c_k|u'(t_k)|
$$

$$
\leq |u_0| + a|u_0'| + M(1 + \|u(t)\|_{E_0}) \int_0^aads + |u(t)||_{E_0} \sum_{0 < \xi < t} (b_k + ac_k)
$$

$$
\leq |u_0| + a|u_0'| + M(1 + \|u(\theta(t))\|) \int_0^aads + |u(\phi(t))| \sum_{k=1}^{p}(b_k + ac_k)
$$

$$
\leq |u_0| + a|u_0'| + a^2M(1 + h) + h \sum_{k=1}^{p}(b_k + ac_k), t \in J,
$$

so

$$
\|Au(\phi(t))\|_{PC} \leq |u_0| + a|u_0'| + a^2M(1 + h) + h \sum_{k=1}^{p}(b_k + ac_k). \tag{3.8}
$$

Similarly, from (3.2), (H1), (H2) and (2.1), we get

$$
\left| \frac{dAu(\phi(t))}{d\phi(t)} \right| \frac{d\phi(t)}{dt} = |(Au(\phi(t)))'| \leq |u_0'| + aM(1 + h) + h \sum_{k=1}^{p}c_k, t \in J,
$$

$$
\left| \frac{dAu(\phi(t))}{d\phi(t)} \right| \leq \frac{1}{m_1} \left[ |u_0'| + aM(1 + h) + h \sum_{k=1}^{p}c_k \right], t \in J,
$$

5
so

\[
\|(Au(\phi(t)))'\|_{PC^*} \leq \frac{m_2}{m_1} [u_0' + aM(1 + h) + h \sum_{k=1}^{p} c_k].
\]  

(3.9)

According to (3.8) and (3.9), we obtain

\[
\|Au(\phi(t))\| \leq \max \left\{ \|u_0\| + a\|u_0'\| + a^2M(1 + h) + h \sum_{k=1}^{p} (b_k + ac_k), \frac{m_2}{m_1} [u_0' + aM(1 + h) + h \sum_{k=1}^{p} c_k] \right\}.
\]

Therefore \(A(T)\) is uniformly bounded.

Step 3. \(A(T)\) is equicontinuous on every \(J_k (k = 0, \cdots, p)\), where \(J_0 = [0, \xi_1], J_k = (\xi_k, \xi_{k+1}] (k = 1, \cdots, p)\).

For any \(Au(\phi(t)) \in A(T)\) and any \( \varepsilon > 0\), take \( \delta = \left[ \|u_0'\| + aM(1 + h) + h \sum_{k=1}^{p} c_k \right]^{-1} \varepsilon. \) Then if \( t_1, t_2 \in J_k \) and \( |t_1 - t_2| < \delta \) with \( t_1 < t_2 \), from (3.1), (H1), (H2) and (2.1), we have

\[
|Au(\phi(t_2)) - Au(\phi(t_1))| \leq \|u_0\|(t_2 - t_1) + \int_{t_1}^{t_2} (t - s)M(1 + \|u(s)\|_{E_0})ds + \sum_{k=1}^{p} (t_2 - t_1)c_k|\phi'(t_1)|
\]

\[
\leq \left[ \|u_0'\| + aM(1 + \|u(t)\|_{E_0}) + \|u(t)\|_{E_0} \sum_{i=1}^{p} c_i \right] |t_2 - t_1| \leq \left[ \|u_0'\| + aM(1 + h) + h \sum_{k=1}^{p} c_k \right] |t_2 - t_1| < \varepsilon.
\]

Thus, \(A(T)\) is equicontinuous on every \(J_k (k = 0, \cdots, p)\).

As a consequence of Step 1-3, \(A\) is completely continuous.

(ii) For any \(\|u(\phi(t))\| \in G\), similar with getting (3.8) and (3.9), we have respectively

\[
\|Au(\phi(t))\|_{PC} \leq \|u_0\| + a\|u_0'\| + a^2M + \left[ a^2M + \sum_{k=1}^{p} (b_k + ac_k) \right] \|u(\phi(t))\|
\]

\[
= \|u_0\| + a\|u_0'\| + a^2M + l_1\|u(\phi(t))\|,
\]

\[
\|(Au(\phi(t))')\|_{PC^*} \leq \frac{m_2}{m_1} \left[ \|u_0'\| + aM \right] + \frac{m_2}{m_1} \left[ aM + \sum_{k=1}^{p} c_k \right] \|u(\phi(t))\| = \frac{m_2}{m_1} \left[ \|u_0'\| + aM \right] + l_2\|u(\phi(t))\|.
\]

Then \(\|u(\phi(t))\| = \lambda\|Au(\phi(t))\| \leq \|Au(\phi(t))\| \leq L + l\|u(\phi(t))\|, \) where \(L = \max \left\{ \|u_0\| + a\|u_0'\| + a^2M, \frac{m_2}{m_1} \left[ \|u_0'\| + aM \right] \right\}. \) It follows that \(\|u(\phi(t))\| \leq \frac{L}{1 - l}, \) i.e., \(G\) is bounded.

From (i) and (ii), now all conditions of Lemma 2.2 are satisfied and therefore the proof is complete.

\[\square\]

**Theorem 3.2.** If conditions (H1) \((I_{0k}(0) = 0, I_{1k}(0) = 0\) are not needed\) and (H4) are satisfied, then (1.1) has a unique solution.

The proof of Theorem 3.2 is similar to that of Theorem 3.1, and is omitted here.

**Remark 3.1.** By comparing Theorem 3.1-3.2, each of them has his own strong and weak points. The condition (H3) of Theorem 3.1 is more easily satisfied than the condition (H4) of Theorem 3.2. The condition (H2) of Theorem 3.1 is also satisfied easily, but Theorem 3.2 hasn’t the condition. The result of Theorem 3.1 determines that (1.1) has at least one solution in the closed ball \(B\).

**Remark 3.2.** If the corresponding formulas of (1.1), (H1), (H3) and (H4) are respectively changed into \(D\Delta u(t_k) = I_0(u(t_k), u'(t_k)), \ Delta u(t_k) = I_1(u(t_k), u'(t_k))\) of (1.1), \(|I_{0k}(x_2(t_k)) - I_{0k}(x_1(t_k))| \leq b_{k1}|x_2(t_k) - x_1(t_k)| + b_{k2}|y_2(t_k) - y_1(t_k)|, |I_{1k}(y_2(t_k)) - I_{1k}(y_1(t_k))| \leq c_{k1}|x_2(t_k) - x_1(t_k)| + c_{k2}|y_2(t_k) - y_1(t_k)|\) of (H1), \(l_1 = a^2M + \sum_{k=1}^{p} (b_{k1} + b_{k2}) + a(c_{k1} + c_{k2}), l_2 = \frac{m_2}{m_1} (aM + \sum_{k=1}^{p} (c_{k1} + c_{k2}))\) of (H3),
Consider the equation
\[ B | M l \]
where \( b \) in the closed ball (H4), then there are also the same results as Theorem 3.1-3.2.

4 Examples

Example 4.1. Consider the equation

\[
\begin{align*}
\left\{ \begin{array}{l}
(\frac{1}{2}t(1-t))'' &= \frac{t}{66} \left[ 11 \sin(u(t) + e^t) - 2u'(t) + 6 \int_0^t (ts)u(s)ds + 3 \int_0^t (ts^2)u(s)ds \right],
\Delta u(t_1) = \frac{1}{12} u(t_1), \quad \Delta u'(t_1) = \frac{1}{12} u'(t_1),
\end{array} \right.
\end{align*}
\]

(4.1)

Firstly, it is easy to verify that \( \phi(t) = t + \frac{1}{2} t(1-t), \) \( k(t, s) = ts, k_0 = 1, h(t, s) = ts^2, h_0 = 1 \) all satisfy the requisitions of (1.1). From \( \phi'(t) = \frac{3}{2} - t, \) we get \( m_1 = 1/2, \) \( m_2 = 3/2. \) Next, since

\[
f(t, x, y, z_1, z_2) = \frac{t}{66} \left[ 11 \sin(x + e^t) - 2y + 6z_1 + 3z_2 \right], \quad \text{and} \quad |\sin(x_2(t) + e^t) - \sin(x_1(t) + e^t)| = |(x_2(t) + e^t) - (x_1(t) + e^t)| \leq |x_2(t) - x_1(t)| \quad (\bar{x}(t) \text{ is located between } x_1(t) \text{ and } x_2(t)),
\]

we have

\[
|f(t, x_2, y_2, z_{12}, z_{22}) - f(t, x_1, y_1, z_{11}, z_{21})| \\
\leq \frac{t}{66} \left[ 11 |\sin(x_2 + e^t) - \sin(x_1 + e^t)| + 2|y_2 - y_1| + 6|z_{12} - z_{11}| + 3|z_{22} - z_{21}| \right] \\
\leq t \left[ \frac{1}{6} |x_2 - x_1| + \frac{1}{3} |y_2 - y_1| + \frac{1}{11} |z_{12} - z_{11}| + \frac{1}{22} |z_{22} - z_{21}| \right] \\
\leq t \left[ \frac{1}{6} |x_2 - x_1| + \frac{1}{3} |y_2 - y_1| + \frac{1}{11} |z_{12} - z_{11}| + \frac{1}{22} |z_{22} - z_{21}| \right], \quad t \in J,
\]

where \( b = \frac{1}{6}, \ c = \frac{1}{33}, \ d_1 = \frac{1}{11}, \ d_2 = \frac{1}{22}, \ a = 1, \ a_0 = \int_0^1 t dt = \frac{1}{2}. \) From \( I_{01}(x) = \frac{1}{12} x, \) \( I_{11}(y) = \frac{1}{12} y, \) we have

\[
|I_{01}(x_2(t_1)) - I_{01}(x_1(t_1))| \leq \frac{1}{12} |x_2(t_1) - x_1(t_1)|, \quad I_{01}(0) = 0, \\
|I_{11}(y_2(t_1)) - I_{11}(y_1(t_1))| \leq \frac{1}{12} |y_2(t_1) - y_1(t_1)|, \quad I_{11}(0) = 0,
\]

where \( b_1 = c_1 = \frac{1}{12}. \) Further, we have

\[
|f(t, u(t), u'(t), Ku(t), Hu(t))| \\
\leq \frac{t}{66} \left[ 11 |\sin(u(t) + e^t)| + 2|u'(t)| + 6 \int_0^t k(t, s)|u(s)|ds + 3 \int_0^t h(t, s)|u(s)|ds \right] \\
\leq \frac{t}{66} \left[ 11 |u(t)|x_0 + 6|u(t)||x_0 + 3\|u(t)\|x_0 \right] = \frac{1}{6} (1 + ||u(t)||x_0),
\]

where \( M = \frac{1}{6}. \) Finally, since \( l_1 = a^2 M + (b_1 + ac_1) = \frac{1}{3}, \) \( l_2 = \frac{m_2}{m_1} (aM + c_1) = \frac{3}{4}. \) we get

\( l = \max\{l_1, l_2\} = \frac{3}{4} < 1. \)

Thus (4.1) satisfies all conditions of Theorem 3.1. It follows that (4.1) has at least one solution in the closed ball \( B. \)
Example 4.2. Consider the equation

\[
\begin{cases}
(u(t + \frac{1}{2} t(1-t)))'' &= \frac{t}{108} \left[ 12 \sqrt{1 + u''(t)} - 6 \arctan(u'(t) + e') + 3 \int_0^1 (ts)u(s)ds + 3 \int_0^1 (ts^2)u(s)ds \right], \quad t \in J = [0, 1], \ t \neq \xi_1 = \frac{1}{2}, \\
\Delta u(t_1) &= \frac{1}{18} u(t_1) + 1, \ \Delta u'(t_1) = \frac{1}{18} u'(t_1) + 2, \ t_1 = \frac{5}{8}, \\
\Delta u(0) &= u_0, \ u'(0) = u_0'.
\end{cases}
\] (4.2)

Firstly, it is easy to verify that \( \phi(t) = t + \frac{1}{2} t(1-t), \ k(t, s) = ts, \ k_0 = 1, \ h(t, s) = ts^2, \ h_0 = 1 \) all satisfy the requisitions of (1.1). From \( \phi'(t) = \frac{3}{2} - t, \) we get \( m_1 = 1/2, m_2 = 3/2. \) Next, since \( f(t, x, y, z_1, z_2) = \frac{t}{108} \left[ 12 \sqrt{1 + x^2} - 6 \arctan(y + e') + 3(2 + z_2) \right], \) and \( |1 + x_2(t) - \sqrt{1 + x_1^2(t)}| = |x_2(t) - x_1(t)|, \ |\arctan(y_2(t) + e') - \arctan(y_1(t) + e')| = |y_2(t) + e') - (y_1(t) + e')|, \)

\[
\frac{1}{1 + (y(t) + e')^2} \leq |y_2(t) - y_1(t)| \ (\hat{x}(t) \text{ is located between } x_1(t) \text{ and } x_2(t), \ \hat{y}(t) \text{ is located between } y_1(t) \text{ and } y_2(t)),
\]

we have

\[
|f(t, x_2, y_2, z_1, z_2) - f(t, x_1, y_1, z_1, z_2)| \\
\leq \frac{t}{108} \left[ 12 \sqrt{1 + x_2^2} - \sqrt{1 + x_1^2} + 6 |\arctan(y_2 + e') - \arctan(y_1 + e')| + 3 \sum_{i=1}^2 |z_{2i} - z_{1i}| \right] \\
\leq \frac{1}{9} |x_2 - x_1| + \frac{1}{18} |y_2 - y_1| + \frac{1}{36} \sum_{i=1}^2 |z_{2i} - z_{1i}| \\
\leq \frac{1}{9} \|x_2 - x_1\|_{PC} + \frac{1}{18} \|y_2 - y_1\|_{PC} + \frac{1}{36} \sum_{i=1}^2 \|z_{2i} - z_{1i}\|_{PC}, \ t \in J,
\]

where \( b = \frac{1}{9}, \ c = \frac{1}{18}, \ d_1 = d_2 = \frac{1}{36}, \ a = 1, \ a_0 = \int_0^1 tdt = \frac{1}{2}. \) From \( I_{01}(x) = \frac{1}{18} x + 1, \ I_{11}(y) = \frac{1}{18} y + 2, \) we have

\[
|I_{01}(x_2(t_1)) - I_{01}(x_1(t_1))| \leq \frac{1}{18} |x_2(t_1) - x_1(t_1)|, \ |I_{11}(y_2(t_1)) - I_{11}(y_1(t_1))| \leq \frac{1}{18} |y_2(t_1) - y_1(t_1)|,
\]

where \( b_1 = c_1 = \frac{1}{18}. \) Finally, since \( r_1 = a_1 = \frac{1}{18} \), we get \( r = \max\{r_1, r_2\} = \frac{1}{2} < 1. \)

Thus (4.2) satisfies all conditions of Theorem 3.2. It follows that (4.2) has a unique solution.

5 Conclusion

We have derived some existence results for second order neutral impulsive integro-differential equations with advanced argument. Firstly, \( u(t) \in E_0 \) is a solution of (1.1) if and only if \( u(t) \in E_0 \) is a solution of the integral equation. Although the methods used are conventional, the impulsive integro-differential equations are different from the past, with that are higher-order and advanced. The difficulty of solving the problem have increased a lot, with the equations are from the first order and advanced to the higher order and advanced. Because Theorem 3.1-3.2 have their own advantages and disadvantages, we can choose according to the conditions given. Finally, two examples are
given to illustrate the effectiveness and superiority of our main results, which are compared with the examples for impulsive differential equations from existing literature.

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Competing Interests
Authors have declared that no competing interests exist.

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