A Generalization of Tychonoff’s Product Theorem

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Author’s contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

A set-theoretical generalization of Tychonoff’s theorem on compactness of the product of compact topological spaces is proved.

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1 Introduction

The original proof of Tychonoff’s product theorem relies on the Kuratowski – Zorn lemma, i.e., on the axiom of choice. Later it was shown (see [1] and the references therein) that this axiom can be eliminated from the proof (but not from the assertion). In [1], the notion of (pointless) topological space was generalized to this end. In the present article, we generalize it yet more (joins are not mentioned at all) and prove in this generality an elementary set-theoretical compactness theorem having no prototypes in the literature. As a topological application of this main result, we give a choice-free proof of the classical Tychonoff theorem.

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2 Preliminaries

A subset of all nonempty finite subsets of a set \( I \) will be denoted by \((2^I)\). The least element, provided it exists, of an ordered set will be denoted by \( 0 \), maybe with a subscript. The symbols \( \inf \) and \( \wedge \) are equipollent.

We will say that, by definition,

- an ordered set is bottomed if it contains the exact lower bound of every its subset;
- a subset (below referred to as a system) \( \{x_i, i \in I\} \) of a bottomed lattice is centered if for any \( J \in (2^I) \) \( \bigwedge_{i \in J} x_i \neq 0 \);
- a unary operation \( f \) in an ordered set \( X \) is: inductive (idempotent) if for any \( x \in X \) \( x \prec f(x) \) \((f(f(x)) = f(x)\), respectively), a hypoclosing if it is both inductive and idempotent;
- the element \( f(x) \), where \( f \) is a hypoclosing, is a hypoclosure of \( x \);
- an element coinciding with its hypoclosure is hypoclosed;
- an equipped with a inductive idempotent operation bottomed set is a hypotopological space;
- an element \( x \) of a hypotopological space is hypocompact if for any centered system \( Z \) of hypoclosed elements of this space
  \[
  \bigwedge_{x \in Z} x \wedge z \neq 0;
  \]
- a set \( H \) of hypoclosed elements in a hypotopological space \( X \) is a hypobase of this space if for any \( x \in X \) there exists \( H_x \subset H \) such that \( x = \bigwedge_{y \in H_x} y \).

**Lemma 2.1.** Let \( U \) be a bottomed set, \( V \) be its bottomed subset, and \( f \) be the operation in \( U \) defined by \( f(u) = \inf\{v \in V : u \prec v\} \). Then \( f \) is idempotent.

**Proof.** Obviously, \( f(u) \prec f(f(u)) \). For \( x \prec f(f(u)) \) and \( w \in V \), the obvious implication \( f(u) \prec w \Rightarrow x \prec w \) shows that \( x \prec f(u) \). Thus \( f(f(u)) \prec f(u) \).

**Proposition 2.1.** In order that an element \( x \) of a hypotopological space \( X \) be hypocompact it is necessary and sufficient that there exist a hypobase \( F \) in \( X \) such that for any centered system \( Z \subset F \) relation (2.1) holds.

**Proof.** Necessity follows from the obvious fact that every hypotopology is its own hypobase.

Sufficiency. Let \( F \) be a hypobase. Then for any hypoclosed element \( y \) there exists \( Z_y \subset F \) such that \( y = \bigwedge_{x \in Z_y} x \). Let \( y \) range over some centered system \( Y \). On the strength of the last equality \( y \prec z \) as \( z \in Z_y \), so the system \( Z \equiv \bigcup_{y \in Y} Z_y \) is also centered. So if for \( x \) and \( F \) are as in the assertion, then relation (2.1) holds. Its left-hand side is none other than \( \bigwedge_{y \in Y} x \wedge y \).

Let \( X \) be a bottomed set, \( \Theta \) be a nonempty set; for each \( \theta \in \Theta \), \( X_\theta \) be a hypotopological space with hypoclosing \( \gamma \) (denoted likewise for all \( \theta \)), and \( p_\theta : X \to X_\theta \) be an operator with the property

\[
(\forall \theta \in \Theta \quad p_\theta y = p_\theta z) \Rightarrow y = z
\]

(i. e., the family \( \{p_\theta, \theta \in \Theta\} \) separates points of \( X \)).

Denote

\[
X_* = \{x \in X : \exists y \in X \quad \forall \theta \in \Theta \quad p_\theta y = \overline{p_\theta x}\}
\]
(≠ ∅ since 0 ∈ X.). If for some x ∈ X, p_θ y = p_θ x for all θ (the element y with this property is unique due to (2.2)), then we put by definition [x]_θ = y. Otherwise speaking, for x ∈ X, the element [x]_θ is uniquely determined by the formula

\[ p_θ[x]_θ = p_θ x, \ θ ∈ Θ. \]

Denote further

\[ H = \{ x ∈ X_*: [x]_* = x \}, \ X = \left\{ \bigwedge_{y ∈ S} y, \ S ⊂ H \right\} \]

and define the operation [·] on X by

\[ [x] = \inf \{ z ∈ X : x ≺ z \} \equiv \bigwedge_{x ≺ z \in X} z. \quad (2.3) \]

Obviously, the set X is bottomed.

**Lemma 2.2.** The operation [·] is a hypoclosing.

*Proof.* Inductiveness is immediate from (2.3), idempotence – from (2.3) and Lemma 2.1. \(\square\)

### 3 The Main Result

In this section, we use the Gothic font for the last three letters of the alphabet and for H. So

\[ X_* = \{ x ∈ X : \exists η ∈ X : ∀ θ ∈ Θ, p_θ η = p_θ x \}; \]

\[ H = \{ x ∈ X : [x]_* = x \}, \ X = \left\{ \bigwedge_{y ∈ S} r, \ S ⊂ H \right\}; \]

for x ∈ X, the element (whose uniqueness will emerge from (3.1)) [x]_θ is defined by the formula

\[ p_θ [x]_θ = p_θ x, \ θ ∈ Θ; \]

the operation [·] is defined by

\[ [x] = \inf \{ z ∈ X : x ≺ z \} \equiv \inf \{ z ∈ H : x ≺ z \}. \]

**Theorem 3.1.** Let the following objects be given: a nonempty set Θ and a bottomed set X; for each θ ∈ Θ – a hypotopological space X_θ with a hypoclosing \(\tau\) (denoted likewise for all θ) and an operator p_θ : X → X_θ; a set \(\mathfrak{J} \subset X\) such that for any θ ∈ Θ \(p_θ \mathfrak{J}\) is a hypobase in X_θ. Assume that the family \(\{ p_θ, θ ∈ Θ \}\) has the properties:

\[ (∀ θ ∈ Θ : p_θ η = p_θ y) ⇒ η = y, \quad (3.1) \]

for any system \(\{ y_i, i ∈ I \} ⊂ X\) there holds

\[ \left( (∀ θ ∈ Θ : \bigwedge_{i ∈ I} p_θ y_i ≠ 0_θ) \right) ⇒ \bigwedge_{i ∈ I} y_i ≠ 0. \quad (3.2) \]

Then: (i) The operation [·] is a hypoclosing. (ii) If \(x\) is an element of X such that for each θ ∈ Θ \(p_θ x\) is hypocompact, then \(x\) is itself hypocompact.
Note prior to the proof that property (3.1) provides correctness of the definition of \([\cdot]_\theta\) (\([\cdot]_\theta\) is unique for any \(\theta \in X_\theta\)).

**Proof.** The first statement is the replica of Lemma 2.2, so we prove the second. Fix \(\theta\). By condition the element \(p_\theta \mathcal{Z}\) is hypocompact. Let \(Z\) be a centered subsystem of \(\mathcal{Z}\). Then so is the system \(p_\theta Z\) (since, for any mapping, the intersection of the images of arbitrary sets contains the image of their intersection), whence by the definition of hypocompactness

\[
\bigwedge_{y \in Z} p_\theta y \wedge p_\theta y \neq 0_\theta,
\]

which together with (3.2) and arbitrariness of \(\theta\) yields

\[
\bigwedge_{y \in Z} f \wedge \mathcal{Z} \neq 0.
\]

It remains to refer to Proposition 2.1. \(\square\)

Note that the proof of the theorem does not rely on the axiom of choice.

4 Tychono’s Theorem as a Consequence of Theorem 3.1

Let \(T\) be an infinite set, and \(\{X_t, t \in T\}\) be a family of nonempty sets. A point of the Cartesian product \(\prod_{t \in T} X_t\) will be denoted by \(x(\cdot)\). It is a mapping of \(T\) into \(\bigcup_{t \in T} X_t\) such that for each \(t \in T\) \(x(t) \in X_t\). Take \((2^T)_0\) as \(\Theta\), the Boolean of \(\prod_{t \in T} X_t\) as \(\mathcal{X}\), the Boolean of \(\prod_{t \in \theta} X_t\) as \(\mathcal{X}_\theta\). The sets \(\mathcal{X}\) and \(\mathcal{X}_0, \theta \in \Theta\), are ordered with \(\subset\) as \(\prec\). Under the above choice of \(\mathcal{X}\), its elements denoted in Section 3 by the lowercase Gothic letters turn out to be subsets of \(\prod_{t \in T} X_t\), so we use for them the customary symbols \(A, B\) etc. Define, for each \(\theta \in \Theta\) and \(A \subset X\), \(p_\theta A\) by

\[
p_\theta A = \{x(\cdot)|_\theta, x(\cdot) \in A\},
\]

so that \(\mathcal{X}_\theta = p_\theta \mathcal{X}\). Condition (3.1) takes the form

\[
\left(\forall \theta \in (2^T)_0 \right) \{x(\cdot)|_\theta, x(\cdot) \in A\} = \{x(\cdot)|_\theta, x(\cdot) \in B\} \Rightarrow A = B.
\]

Let us prove even the stronger relation

\[
\left(\forall \theta \in (2^T)_0 \right) \{x(\cdot)|_\theta, x(\cdot) \in A\} \subset \{x(\cdot)|_\theta, x(\cdot) \in B\} \Rightarrow A \subset B.
\]

If the antecedent is true, then one can write the obvious implications

\[
x(\cdot) \notin B \Rightarrow \exists s \in T s x(s) \notin p_\theta B \Rightarrow \exists \theta \in (2^T)_0 x(\cdot)|_\theta \notin p_\theta A \Rightarrow x(\cdot) \notin A.
\]

Condition (3.2) takes the form

\[
\left(\forall \theta \in (2^T)_0 \right) \bigcap_{i \in T} p_\theta B_i \neq \emptyset \Rightarrow \bigcap_{i \in T} B_i \neq \emptyset.
\]
Here is the proof of this relation. Let us take, for each $s \in T$, a point $x_s \in \bigcap_{i \in I} p(s) B_i$ and define $x(\cdot)$ by $x(s) = x_s$. By construction $x(\cdot)|_{\{s\}} = x_s$, so for all $s \in T x(\cdot)|_{\{s\}} \in p(s) B_i$, $i \in I$, and therefore $x(\cdot) \in B_i$, $i \in I$.

From this time on, the $X_i$'s are topological spaces, and $\tau$ is the topological closing in every $\prod_{\theta \in \Theta} X_{\theta}$, as well as in their Tychono’s product. For $E \in X_i$, $E^c$ and $C_i(E)$ signify $X_i \setminus E$ and $\{x(\cdot) \in \prod_{i \in T} X_i : x(t) \in E\}$, respectively. For a subset $E$ either of $X_i$ or of $\prod_{i \in T} X_i$, $E^c$ means the interior of $E$.

Lemma 4.1. Let, for each $t \in T$, $F_t$ be a closed set in a topological space $X_t$. Then $\prod_{t \in T} F_t$ is a closed set in $\prod_{t \in T} X_t$ equipped with Tychono’s topology.

Proof. $\prod X_t \setminus \prod F_t = \bigcup C_i(F_t^c)$.

The next statement is obvious.

Lemma 4.2. For any $t \in T$ and $B_t \subseteq X_t$ the set $C_i(B_t)$ is open iff so is $B_t$.

Corollary 4.3. For any $t \in T$ and $B_t \subseteq X_t$ $(C_i(B_t))^c \subseteq C_i(B_t^c)$.

Proof. The left-hand side is the union of all open subsets of $C_i(B_t)$, and the collection of the latter is, by Lemma 4.2, exhausted by sets of the kind $C_i(G_t)$, where $G_t$ is an open subset of $B_t$.

Lemma 4.4. Let, for each $t \in T$, $A_t$ be a set in a topological space $X_t$. Then $\prod_{t \in T} A_t = \prod_{t \in T} A_t^c$.

Proof. Denote $Q = \prod_{t \in T} A_t$, $R = \prod_{t \in T} A_t^c$. In this proof, the superscript $c$ will be written after the symbols of subsets of $\prod X_t$, as well, and will signify the complement to $\prod X_t$. One has $Q \subseteq R$ since $Q \subseteq R$ and $R$ is closed by Lemma 4.1. On the other hand, $R^c = \bigcup C_i(A_t^c) = \bigcup C_i(A_t^c)^c$,

$Q^c = Q^c = (C_i(A_t^c))^c \supset \bigcup(C_i(A_t^c))^c$.

It remains to apply Corollary 4.3 to $B_t = A_t^c$.

Denote

$$X_* = \left\{ \prod_{k=1}^m D_{kt} : m \in \mathbb{N}, D_{kt} \subset X_t \right\}.$$ 

Obviously, $p_0 \prod_{k=1}^m D_{kt} = \prod_{k=1}^m D_{kt}$. Hence we get by Lemma 4.4

$$p_0 \prod_{k=1}^m D_{kt} = \bigcup_{k=1}^m D_{kt} \equiv p_0 \prod_{k=1}^m D_{kt}.$$ 

So, for

$$\tau = \bigcup_{k=1}^m D_{kt}.$$
one has

\[ \langle x \rangle_* = \bigcup_{k=1}^{m} \prod_{t \in T} D_{kt}. \]

Put

\[ Z = \left\{ \bigcup_{k=1}^{m} \prod_{t \in T} F_{kt} : m \in \mathbb{N}, F_{kt} \subset X_t, \text{ the } F_{kt}'s \text{ are closed} \right\}. \]

Let us show that for each \( \theta \in \Theta \) \( p_{\theta} Z \) is a hypobase in \( X_\theta \). To this end we will show even more: \( \{ Q^\theta : Q \in p_{\theta} Z \} \) is a base in \( X_\theta \). Obviously,

\[ p_{\theta} Z = \left\{ \bigcup_{k=1}^{m} \prod_{t \in \Theta} F_{kt} : m \in \mathbb{N}, F_{kt} \subset X_t, \text{ the } F_{kt}'s \text{ are closed} \right\}, \]

\[ \prod_{t \in \Theta} X_t \setminus \bigcup_{k=1}^{m} \prod_{t \in \Theta} F_{kt} = \bigcup_{k=1}^{m} p_{\theta} C_t (F_{kt}). \]

The set \( F_{kt}^0 \) being open, so is the set on the right-hand side. Writing, for arbitrary \( R_{kt} \subset X_t \) (\( k \in \{1, \ldots, m\}, t \in \Theta \)), the identity

\[ \bigcup_{k=1}^{m} \prod_{t \in \Theta} R_{kt} = \bigcup_{j(\cdot) \in \Theta^m} \bigcap_{k=1}^{m} R_{k f(j)}, \]

we see that the sets \( \bigcap_{k=1}^{m} \bigcup_{T \in \Theta} F_{kt} \cap (G_{kt}) \) create a base of \( X_{\theta} \) since so do even the sets \( \bigcap_{k=1}^{m} p_{\theta} C_t (G_{kt}) \).

Obviously, the hypotopological space \( X \) with the above hypobase \( Z \) is the same as the topological space \( X \) with the Tychonoff’s topology. Herein, for a topological space with closed sets in the role of hypoclosed ones, hypocompactness is tantamount to compactness. Thus we have deduced from Theorem 3.1 the following intermediate statement on the way to Tychonoff’s theorem.

**Lemma 4.5.** Let \( X_t, t \in T, \) be compact topological spaces such that for any \( \theta \in (2^T)_0 \) \( \prod_{t \in \Theta} X_t \) is compact. Then \( \prod_{t \in T} X_t \) is compact in the Tychonoff topology.

Applying this statement to \( T = \{1, 2\} \), we get

**Corollary 4.6.** The Tychonoff product of two compact topological spaces is compact.

Hence we deduce by induction

**Corollary 4.7.** The Tychonoff product of a finite number of compact topological spaces is compact.

Juxtaposing Lemma 4.4 with Corollary 4.6, we arrive at the theorem of Tychonoff [2, 3]: The Tychonoff product of compact topological spaces is itself compact. Note that the deduction does not rely on the axiom of choice. In fact, this axiom underlies only the assertion of the theorem in its general form where it postulates nonemptiness of the Cartesian product. But in the cases when all the \( X_t \)'s coincide or all of them are closed intervals the axiom is unnecessary even in the assertion.

In the proof of the Banach – Alaoglu theorem [3, Th. 3.15], just the second situation occurs. So that theorem turns out independent of the axiom of choice.

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Competing Interests
Author has declared that no competing interests exist.

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