An Approximate Solution for a Nonlinear Duffing – Harmonic Oscillator

Van Hieu – Dang

1Department of Mechanics, Thai Nguyen University of Technology, Thai Nguyen, Viet Nam.

Author’s contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/ARJOM/2019/v15i430154

(1) Dr. Xingting Wang, Department of Mathematics, Howard University, Washington DC, USA.

Reviewers:

(1) Chunhua, Alabama State University, USA.

(2) Mohamed A. El-Beltagy, Cairo University, Egypt.

(3) Adel H. Phillips, Ain-Shams University, Egypt.

Complete Peer review History: http://www.sciarticle4.com/review-history/52367

Received: 20 August 2019
Accepted: 28 October 2019
Published: 05 November 2019

Abstract

The equivalent linearization method introduced by Caughey is a powerful tool for analyzing random oscillations. The method is also easy to apply for deterministic oscillations. However, with strong nonlinear systems, the error of this method is usually quite large and even not acceptable. In conjunction with a weighted averaging, the equivalent linearization method has shown more accuracy than the classical one in which the conventional averaging value is used. Combining advantages of the classical equivalent linearization method and accuracy of the weighted averaging, the proposed method has shown that it is a useful tool for analyzing nonlinear oscillations including strong nonlinear systems. In this paper, the proposed method is applied to analyze a nonlinear Duffing – harmonic oscillator. The present results are compared with the results obtained by using other analytical methods, exact results and numerical results.

Keywords: Equivalent linearization method; weighted averaging; nonlinear oscillators.

1 Introduction

Oscillation is a common phenomenon in nature and engineering. In engineering, oscillations occur in many fields, such as oscillation of machinery in mechanics, oscillation of buildings or bridges in the field of

*Corresponding author: E-mail: hieudv@tnut.edu.vn;
construction, oscillation of traffic vehicles, or oscillation of circuits, and so on. Each oscillation phenomenon is often described mathematically by nonlinear differential equations. It is difficult to find exact solutions of these equations. Numerical methods have given us very effective tools to describe these oscillations. However, the frequency-amplitude relationship, the most important feature of oscillation problems, is very difficult to find by using numerical methods. Thus, approximate analytical methods are needed to analyze response of nonlinear oscillations. Some approximate analytical methods have been introduced recently, such as the Parameter Perturbation method (PPM) [1], the Variational Iteration Method (VIM) [2], the Homotopy Perturbation Method (HPM) [3], the Energy Balance Method (EBM) [4], the Parameter Expansion Method (PEM) [5], the Variational Approach (VA) [6], the Hamiltonian Approach (HA) [7]; the Homotopy Analysis Method (HAM) [8], the Amplitude-Frequency Formulation (AFF) [9,10] and the Equivalent Linearization method (ELM) [11]. These approximate analytical methods are effectively analytical tools for investigating nonlinear oscillation problems.

The Equivalent Linearization method was proposed by Caughey [11] in 1963 for purpose of analyzing random nonlinear oscillations. This method is based on the classical ELM of Kryloff and Bogoliubov for deterministic nonlinear systems [12]. ELM [11] is used very commonly for analyzing random nonlinear systems. However, this method is only effective for weak nonlinear systems. There were some developments of this method such as the non-parametric ELM based on first-order reliability method [13], the tail-equivalent linearization method [14], ELM using Gaussian mixture [15] and the regulated Gaussian ELM [16, 17].

In a conventional approach of ELM, the nonlinear system is replaced by an equivalent linear system in which the parameters of the equivalent linear system are determined by minimizing a measure of the discrepancy between the responses of the nonlinear and linear systems called the mean-square criterion [11]. To improve the accuracy of ELM, some extensions of the mean-square criterion was proposed such as a dual criterion [18,19] and local mean square error criterion [20]. Recently, Anh et al. [21,22] introduced a new development of ELM based on first-order reliability method [13], the tail-equivalent linearization method [14], ELM using Gaussian mixture [15] and the regulated Gaussian ELM [16, 17].

In this paper, we consider a nonlinear Duffing-harmonic oscillator given as [26]:

\[ \ddot{u} + k_1 u + \frac{k_3 u^3}{1 + k_2 u^2} = 0, \]  
(1)

with the initial conditions:

\[ u(0) = A, \quad \dot{u}(0) = 0. \]  
(2)

where \( k_1, k_2 \) and \( k_3 \) are constants.

The equivalent linearization method will be applied to find the approximate analytical solution of Eq. (1). First, we rewrite Eq. (1) in the following form:

\[ (1 + k_2 u^2)\ddot{u} + k_1 u + k_2 k_3 u^3 + k_3 u^3 = 0. \]  
(3)
We introduce the linear form of the nonlinear Eq. (1) as follows:

$$\ddot{u} + \omega^2 u = 0.$$  \hspace{1cm} (4)

where $\omega$ is known as the approximate frequency of oscillation and needs to be determined. The coefficient $\omega$ can be found by many different criteria, in which the most common criterion is the mean square criterion requiring the mean square of equation error between the nonlinear equation (3) and the linear equation (4) be minimum [11]:

$$\left\langle e^2(u) \right\rangle = \left\langle \left( k_2 \dot{u}^2 + k_1 u + (k_1 k_2 + k_3) u^3 - \omega^2 u \right)^2 \right\rangle \rightarrow \text{Min}.$$  \hspace{1cm} (5)

where $\langle \rangle$ is the averaging operator; the mean square criterion (5) states that when the error between the nonlinear equation (3) and the linear one (4) is minimum, the solution of the linear equation will be closest to the solution of the nonlinear equation. From the condition $\frac{\partial }{\partial \omega^2} \left\langle e^2(u) \right\rangle$, we get an equation for determining the coefficient $\omega^2$ as follows:

$$k_2 \left\langle \dot{u}^2 \right\rangle + k_1 \left\langle u^2 \right\rangle + (k_1 k_2 + k_3) \left\langle u^4 \right\rangle - \omega^2 \left\langle u^2 \right\rangle = 0.$$  \hspace{1cm} (6)

The periodic solution of linear Eq. (4) with the initial conditions (2) has the form:

$$u = A \cos(\omega t).$$  \hspace{1cm} (7)

The classical averaging value of a harmonic function can be get as [12]:

$$\left\langle f(u(t)) \right\rangle_c = \left\langle f(A \cos(\omega t)) \right\rangle_c = \frac{1}{T} \int_0^T f(A \cos(\omega t))dt = \frac{1}{2\pi} \int_0^{2\pi} f(A \cos(\tau))d\tau$$  \hspace{1cm} (8)

here $T = 2\pi / \omega$ is the period of oscillation and $\tau = \omega t$. The averaging values in Eq. (8) is called the classical averaging value which often leads to solutions of strong nonlinear systems with large and sometimes unacceptable errors. In order to improve this issue, in this paper, a new method is proposed to determine the averaging values in Eq. (6) called weighted averaging value. As proposed by Anh et al. [21, 22], the averaging value can be determined by:

$$\left\langle f(t) \right\rangle_w = \int_0^d h(t) f(t)dt,$$  \hspace{1cm} (9)

where $h(t)$ is the weighted coefficient function satisfying the following condition [21, 22]:

$$\int_0^d h(t)dt = 1.$$  \hspace{1cm} (10)

A specific weighted coefficient function will be used in this paper as follows:

$$h(t) = s^2 \omega^2 te^{-\tau_0 \omega},$$  \hspace{1cm} (11)
herein, \(s\) is a positive constant. We can easily check that \(h(t)\) given in Eq. (11) satisfies the condition (10) as \(d \to \infty\). Based on the weighted coefficient function \(h(t)\) given in Eq. (11), the weighted averaging value of a function \(f(\omega t)\) can be calculated by using Eq. (9):

\[
\langle f(\omega t) \rangle_w = \int_0^\infty s^2 \omega^2 e^{-st} f(\omega t) dt = \int_0^\infty s^2 \tau e^{-st} f(\tau) d\tau,
\]

We can see that the weighted averaging value of the function \(f(\omega t)\) is the Laplace transform of function \(s^2 \omega^2 f(t)\) as follows:

\[
\langle f(\omega t) \rangle_w = \int_0^\infty s^2 \tau e^{-st} f(\tau) d\tau = F(s).
\]

The averaging value of function \(f(\omega t)\) does not depend on the time \(t\) but depends on the parameter \(s\) (in expression of the weighted coefficient function \(h(t)\)). The parameter \(s\) is called the adjustment parameter, accuracy of the obtained solution depends on the choice of value of the parameter \(s\). The choice of value of the parameter \(s\) has been examined by authors in many cases [22-25], the obtained results will be very accurate when \(s = 2\).

Based on the periodic solution in Eq. (7), we can calculate averaging operators in Eq. (6) by using Eqs. (12) and (13):

\[
\langle u^2 \rangle_w = \langle A^2 \cos^2(\omega t) \rangle_w = \int_0^\infty A^2 s^2 \omega^2 e^{-st} \cos^2(\omega t) dt = \int_0^\infty A^2 s^2 \tau e^{-st} \cos^2(\tau) d\tau
\]

\[
= A^2 \frac{s^4 + 2s^2 + 8}{(s^2 + 4)^2},
\]

\[
\langle u^4 \rangle_w = \langle A^4 \cos^4(\omega t) \rangle_w = \int_0^\infty A^4 s^2 \omega^2 e^{-st} \cos^4(\omega t) dt = \int_0^\infty A^4 s^2 \tau e^{-st} \cos^4(\tau) d\tau
\]

\[
= A^4 \frac{s^8 + 28s^6 + 248s^4 + 416s^2 + 1536}{(s^2 + 4)^2(s^2 + 16)^2},
\]

\[
\langle u^1u \rangle = -\langle A^4 \omega^2 \cos^4(\omega t) \rangle = -\int_0^\infty A^4 \omega^2 s^2 \omega^2 e^{-st} \cos^4(\omega t) dt
\]

\[
= -\int_0^\infty A^4 \omega^2 s^2 \tau e^{-st} \cos^4(\tau) d\tau = -A^4 \omega^2 \frac{s^8 + 28s^6 + 248s^4 + 416s^2 + 1536}{(s^2 + 4)^2(s^2 + 16)^2}.
\]

Substituting Eqs. (14)-(16) into Eq. (6), we obtain the approximate frequency of oscillation:

\[
\omega = \sqrt{\frac{k_1(s^4 + 2s^2 + 8)(s^2 + 16)^2 + (k_2k_3 + k_2)(s^8 + 28s^6 + 248s^4 + 416s^2 + 1536)A^3}{(s^4 + 2s^2 + 8)(s^2 + 16)^2 + k_2(s^8 + 28s^6 + 248s^4 + 416s^2 + 1536)A^3}}.
\]
When $s=2$, we have the approximate frequency:

$$\omega = \sqrt{\frac{k_1 + 0.72(k_2 + k_3)A^2}{1 + 0.72k_2A^2}}. \quad (18)$$

Thus, from Eq. (7), the approximate solution of oscillation can be get as follows:

$$u(t) = A \cos \left( \sqrt{\frac{k_1 + 0.72(k_2 + k_3)A^2}{1 + 0.72k_2A^2}} \right)t. \quad (19)$$

To show accuracy of the present results, we compare the present solution given in Eq. (19) with the numerical one using the 4th-order Runge-Kutta method, the result is presented in Fig. 1 for $k_1=20$, $k_2=100$ and $k_3=20$.

![Fig. 1. Comparison between the present solution with the numerical solution](image)

For purpose of comparing the current solution with published solutions using different approximate analytical methods, we consider some following cases:

**2.1 Case 1: k<sub>2</sub>=0**

If $k_2=0$, from Eq. (1) we have a cubic Duffing oscillator:

$$\ddot{u} + k_1u + k_3u^3 = 0. \quad (20)$$

Eq. (20) was analyzed by authors in ref. [24]. From Eqs. (18) and (19), we can get the approximate frequency:

$$\omega = \sqrt{k_1 + 0.72k_3A^2}, \quad (21)$$

and the approximate solution of oscillation as follows:

$$u(t) = A \cos \left( \sqrt{k_1 + 0.72k_3A^2} \right)t. \quad (22)$$
The approximate frequency of Eq. (20) can be achieved by using the Parametrized Perturbation method (PPM) [27] and the Energy Balance method (EBM) [28] as follows:

$$\omega_{PPM} = \omega_{EBM} = \sqrt{k_1 + 0.75k_3A^2}. \quad (23)$$

The exact frequency of this oscillation is given by [27]:

$$\omega_{exact} = \frac{2\pi}{4\sqrt{2} \int_0^{\pi/2} \frac{dt}{\sqrt{k_3A^2(1 + \cos^2(t)) + 2k_1}}}. \quad (24)$$

Accuracy of the present solution is shown in Table 1 and Fig. 2. For some values of the initial amplitude ($A$) and the coefficients of system ($k_1$ and $k_3$), values of approximate frequencies and exact frequency are showed in Table 1. We can see that the relative errors of the EBM and PPM frequencies reach to 2.2% while the relative error of the current frequency is only 0.15% when the initial amplitude of oscillation increases. With the fixed coefficients $k_1=20$, $k_3=20$ and the initial amplitude $A=5$, time history and phase portrait of approximate solutions and exact solution of the cubic Duffing oscillation are presented in Fig. 2.

![Time history and phase portrait of the cubic Duffing oscillation](image)

**Table 1. Comparison of the approximate frequencies with the exact frequency, case 1**

<table>
<thead>
<tr>
<th>$A$</th>
<th>$k_1$</th>
<th>$k_3$</th>
<th>$\omega_{exact}$ [27]</th>
<th>$\omega_{exact}$ [27, 28]</th>
<th>$R$. error (%)</th>
<th>$\omega_{present}$</th>
<th>$R$. error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1</td>
<td>1</td>
<td>1.0037</td>
<td>1.0037</td>
<td>0.0001</td>
<td>1.0035</td>
<td>0.0147</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>10</td>
<td>3.1741</td>
<td>3.1741</td>
<td>0.0001</td>
<td>3.1736</td>
<td>0.0147</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>100</td>
<td>10.0374</td>
<td>10.0374</td>
<td>0.0001</td>
<td>10.0359</td>
<td>0.0147</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3.1777</td>
<td>3.2228</td>
<td>0.3869</td>
<td>3.1314</td>
<td>0.4771</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>10</td>
<td>4.1671</td>
<td>4.1833</td>
<td>0.3869</td>
<td>4.1472</td>
<td>0.4771</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>100</td>
<td>13.1777</td>
<td>13.2287</td>
<td>0.3869</td>
<td>13.1148</td>
<td>0.4771</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>1</td>
<td>8.5335</td>
<td>8.7177</td>
<td>2.1586</td>
<td>8.5440</td>
<td>0.1220</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>10</td>
<td>26.9855</td>
<td>27.5680</td>
<td>2.1586</td>
<td>27.0185</td>
<td>0.1220</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>100</td>
<td>85.3358</td>
<td>87.1779</td>
<td>2.1586</td>
<td>85.4400</td>
<td>0.1220</td>
</tr>
<tr>
<td>50</td>
<td>1</td>
<td>1</td>
<td>42.3729</td>
<td>43.3128</td>
<td>2.2179</td>
<td>42.4381</td>
<td>0.1538</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>10</td>
<td>133.9951</td>
<td>136.9671</td>
<td>2.2179</td>
<td>134.2013</td>
<td>0.1538</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>100</td>
<td>423.7299</td>
<td>433.1281</td>
<td>2.2179</td>
<td>424.3819</td>
<td>0.1538</td>
</tr>
</tbody>
</table>
2.2 Case 2: \(k_f=0\)

From Eq. (1), we have a Duffing oscillator with a rational elastic term:

\[
\ddot{u} + \frac{k_1 u^3}{1 + k_2 u^2} = 0.
\]  

(25)

From Eq. (18), the approximate frequency can be get as:

\[
\omega = \sqrt{\frac{0.72k_r A^2}{1 + 0.72k_r A^2}}.
\]  

(26)

and from Eq. (19), we can obtain the approximate solution of oscillation:

\[
u(t) = A \cos \left( \sqrt{\frac{0.72k_r A^2}{1 + 0.72k_r A^2}} t \right).
\]  

(27)

With \(k_r=1\) and \(k_f=1\), the approximate frequency of oscillation given in Eq. (25) was achieved by using the Parameter Expansion method (PEM) [27], and the exact frequency of oscillation was given in ref. [29].

Table 2 shows comparison of the present frequency and the PEM frequency with the exact frequency for some values of the initial amplitude \(A\). From Table 2, we can see accuracy of the present solution.

Responses of the oscillation obtained by using different analytical methods and the numerical method are showed in Fig. 3 with \(k_f=20\) and \(k_r=20\).

Table 2. Comparison of the approximate frequencies with the exact frequency, case 2

<table>
<thead>
<tr>
<th>(A)</th>
<th>(\omega_{exact} [29])</th>
<th>(\omega_{PEM} [27])</th>
<th>R. error (%)</th>
<th>(\omega_{present})</th>
<th>R. error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.0085</td>
<td>0.0087</td>
<td>2.2432</td>
<td>0.0085</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.05</td>
<td>0.0423</td>
<td>0.0432</td>
<td>2.2212</td>
<td>0.0424</td>
<td>0.1654</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0844</td>
<td>0.0863</td>
<td>2.2396</td>
<td>0.0846</td>
<td>0.1896</td>
</tr>
<tr>
<td>0.5</td>
<td>0.3874</td>
<td>0.3974</td>
<td>2.5789</td>
<td>0.3906</td>
<td>0.8826</td>
</tr>
<tr>
<td>1</td>
<td>0.6368</td>
<td>0.6547</td>
<td>2.8063</td>
<td>0.6469</td>
<td>1.6034</td>
</tr>
<tr>
<td>5</td>
<td>0.9669</td>
<td>0.9744</td>
<td>0.7622</td>
<td>0.9733</td>
<td>0.6567</td>
</tr>
<tr>
<td>10</td>
<td>0.9909</td>
<td>0.9934</td>
<td>0.2493</td>
<td>0.9931</td>
<td>0.2230</td>
</tr>
</tbody>
</table>

Fig. 3. Time history and phase portrait of the Duffing oscillation with rational elastic term
2.3 Case 3: $k_1=-1$, $k_2=0$ and $k_3=1$

We have a Duffing oscillator with double-well potential (negative linear stiffness):

$$\ddot{u} - u + u^3 = 0. \tag{28}$$

System (28) has three equilibrium points: the central equilibrium point $u = 0$ is unstable and the other two equilibrium points $u = \pm 1$ are stable. The periodic solutions of this oscillation depend on the initial amplitude $A$. For the case of $0 < A < 1$ and $1 < A < \sqrt{2}$, the equilibrium points $u = \pm 1$ are stable. And, for the case of $A > \sqrt{2}$, the periodic solution is symmetric and extends across three equilibrium points.

For the case of $A > \sqrt{2}$, from Eq. (18), the approximate frequency of oscillation can be obtained as:

$$\omega = \sqrt{-1 + 0.72k_3A^2}. \tag{29}$$

The present period $T_{\text{present}}$, the approximate period achieved by the Energy Balance method $T_{\text{EBM}}$ [26] and the exact period $T_{\text{exact}}$ [30] are listed in Table 3 for some values of the initial amplitude ($A$). The exact solution [30], the approximate solution achieved by the Energy Balance method [26] and the present solution are plotted in Fig. 4 for the initial amplitude $A=1.5$. Again we can see accuracy of the present solution.

<table>
<thead>
<tr>
<th>$A$</th>
<th>$T_{\text{exact}}$ [30]</th>
<th>$T_{\text{EBM}}$ [26]</th>
<th>R. error (%)</th>
<th>$T_{\text{present}}$</th>
<th>R. error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.42</td>
<td>15.0844</td>
<td>8.7784</td>
<td>41.8047</td>
<td>9.3477</td>
<td>38.0306</td>
</tr>
<tr>
<td>1.45</td>
<td>11.2132</td>
<td>8.2725</td>
<td>26.2253</td>
<td>8.7656</td>
<td>21.8278</td>
</tr>
<tr>
<td>1.5</td>
<td>9.2237</td>
<td>7.5778</td>
<td>17.8442</td>
<td>7.9797</td>
<td>13.4869</td>
</tr>
<tr>
<td>1.7</td>
<td>6.3528</td>
<td>5.8150</td>
<td>8.4655</td>
<td>6.0438</td>
<td>4.8639</td>
</tr>
<tr>
<td>2</td>
<td>4.6857</td>
<td>4.4429</td>
<td>5.1817</td>
<td>4.5825</td>
<td>2.2024</td>
</tr>
<tr>
<td>5</td>
<td>1.5286</td>
<td>1.4914</td>
<td>2.4335</td>
<td>1.5293</td>
<td>0.3074</td>
</tr>
<tr>
<td>10</td>
<td>0.7471</td>
<td>0.7304</td>
<td>2.2353</td>
<td>0.7457</td>
<td>0.1873</td>
</tr>
<tr>
<td>50</td>
<td>0.1484</td>
<td>0.1451</td>
<td>2.2237</td>
<td>0.1481</td>
<td>0.2021</td>
</tr>
<tr>
<td>100</td>
<td>0.0742</td>
<td>0.0726</td>
<td>2.1563</td>
<td>0.0741</td>
<td>0.1347</td>
</tr>
<tr>
<td>100</td>
<td>0.0074</td>
<td>0.0073</td>
<td>1.3513</td>
<td>0.0074</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Fig. 4. Time history and phase portrait of responds of the Duffing oscillation with double-well potential, $A=1.5$
For the case of $1 < A < \sqrt{2}$, we introduce a new variable:

$$x = u - 1.$$  

(30)

Substituting Eq. (30) into Eq. (28), we get:

$$\ddot{x} + 2x + 3x^2 + x^3 = 0,$$  

(31)

with the initial conditions:

$$x(0) = \tilde{A}, \quad \dot{x}(0) = 0,$$  

(32)

where $\tilde{A} = A - 1$. We will find the approximate solution of Eq. (31), the linear equation of Eq. (31) has the form:

$$\ddot{x} + \omega^2 x = 0.$$  

(33)

Employing the mean square criterion, the coefficient $\omega^2$ can be get as:

$$\omega^2 = \frac{2\langle x^2 \rangle + 3\langle x^3 \rangle + \langle x^4 \rangle}{\langle x^2 \rangle}.$$  

(34)

Using the weighted averaging value, averaging operators in Eq. (34) can be obtained as:

$$\langle x^2 \rangle_w = \tilde{A}^2 \frac{s^4 + 2s^2 + 8}{(s^2 + 4)^2}.$$  

(35)

$$\langle x^3 \rangle_w = \tilde{A}^3 \frac{s^6 + 11s^4 + 43s^2 - 63}{(s^2 + 1)^2(s^2 + 9)^2}.$$  

(36)

$$\langle x^4 \rangle_w = \tilde{A}^4 \frac{s^8 + 28s^6 + 248s^4 + 416s^2 + 1536}{(s^2 + 4)^2(s^2 + 16)^2}.$$  

(37)

Substituting Eqs. (35)-(37) into Eq. (34) and with $\delta=2$, we get the approximate frequency of oscillation for this case:

$$\omega = \sqrt{2 + 1.9824\tilde{A} + 0.72\tilde{A}^2}.$$  

(38)

Note that $\tilde{A} = A - 1$, from Eq. (38), the approximate frequency is given as:

$$\omega = \sqrt{2 + 1.9824(A-1) + 0.72(A-1)^2}.$$  

(39)

Thus, the approximate solution for this case can be get:

$$u(t) = (A-1)\cos\left(\sqrt{2 + 1.9824(A-1) + 0.72(A-1)^2}t\right) + 1.$$  

(40)
To show accuracy of the obtained solution, the exact period $T_{\text{exact}}$ [30], the approximate period obtained by the Energy Balance method $T_{\text{EBM}}$ [26] and the present period $T_{\text{present}}$ are showed in Table 4 for some values of the initial amplitude $A$. We can conclude that the proposed method gives more excellent approximate periods than the EBM for the oscillation amplitude $1 < A < \sqrt{2}$.

Table 4. Comparison of the approximate periods with the exact period for double-well Duffing oscillation ($1 < A < \sqrt{2}$)

<table>
<thead>
<tr>
<th>$A$</th>
<th>$T_{\text{exact}}$ [30]</th>
<th>$T_{\text{EBM}}$ [26]</th>
<th>R. error (%)</th>
<th>$T_{\text{present}}$</th>
<th>R. error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.05</td>
<td>4.3061</td>
<td>4.3045</td>
<td>0.0373</td>
<td>4.3349</td>
<td>0.0067</td>
</tr>
<tr>
<td>1.1</td>
<td>4.1781</td>
<td>4.1748</td>
<td>0.0781</td>
<td>4.2309</td>
<td>0.0126</td>
</tr>
<tr>
<td>1.15</td>
<td>4.0582</td>
<td>4.0530</td>
<td>0.1267</td>
<td>4.1309</td>
<td>0.0179</td>
</tr>
<tr>
<td>1.2</td>
<td>3.9460</td>
<td>3.9384</td>
<td>0.1923</td>
<td>4.0347</td>
<td>0.0225</td>
</tr>
<tr>
<td>1.25</td>
<td>3.8417</td>
<td>3.8303</td>
<td>0.2961</td>
<td>3.9420</td>
<td>0.0261</td>
</tr>
<tr>
<td>1.3</td>
<td>3.7468</td>
<td>3.7282</td>
<td>0.4964</td>
<td>3.8529</td>
<td>0.0283</td>
</tr>
<tr>
<td>1.35</td>
<td>3.6688</td>
<td>3.6316</td>
<td>1.0139</td>
<td>3.7671</td>
<td>0.0268</td>
</tr>
<tr>
<td>1.4</td>
<td>3.6897</td>
<td>3.5399</td>
<td>4.0576</td>
<td>3.6845</td>
<td>0.0014</td>
</tr>
<tr>
<td>1.41</td>
<td>3.8506</td>
<td>3.5222</td>
<td>8.5261</td>
<td>3.6684</td>
<td>0.0473</td>
</tr>
<tr>
<td>1.412</td>
<td>3.9755</td>
<td>3.5164</td>
<td>11.548</td>
<td>3.6652</td>
<td>0.0781</td>
</tr>
</tbody>
</table>

Fig. 5 shows comparison of the present solution and the EBM solution with the exact solution for the initial amplitude $A=1.4$.

![Time history and phase portrait of responds of the Duffing oscillation with double-well potential, $A=1.4$](image)

2.4 Case 4: $k_2=1$

With $k_2=1$, Eq. (1) becomes:

$$\ddot{u} + k_1u + \frac{k_2u^3}{1+u^2} = 0. \tag{41}$$

From Eq. (18), the approximate frequency of oscillation given by Eq. (41) can be obtained as:

$$\omega = \sqrt{\frac{k_1 + 0.72(k_1 + k_2)A^2}{1 + 0.72A^2}}. \tag{42}$$
The approximate frequency of oscillation for this case can be achieved by using the Energy Balance method [26] as follows:

$$\omega_{EBM} = \sqrt{(k_1 + k_3) + 2\frac{k_3}{A^2} \ln \left(1 + \frac{A^2}{2}\right) - \ln(1 + A^2)},$$

(43)

Comparisons of the present solution and EBM solution for this oscillation are presented in Table 5 and Fig. 6. We can see very good agreement of two approximate solutions. Fig. 6 is plotted with $k_1=20$ and $k_3=20$.

<table>
<thead>
<tr>
<th>$k_1$</th>
<th>$k_3$</th>
<th>$A$</th>
<th>$\omega_{EBM}$ [26]</th>
<th>$\omega_{present}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.1</td>
<td>1.0037</td>
<td>1.0037</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1.1936</td>
<td>1.1910</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>5</td>
<td>1.3956</td>
<td>1.3955</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>10</td>
<td>1.4094</td>
<td>1.4094</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>50</td>
<td>1.4140</td>
<td>1.4140</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>100</td>
<td>1.4142</td>
<td>1.4142</td>
</tr>
<tr>
<td>1</td>
<td>100</td>
<td>0.1</td>
<td>1.3207</td>
<td>1.3095</td>
</tr>
<tr>
<td>1</td>
<td>100</td>
<td>1</td>
<td>6.5927</td>
<td>6.5468</td>
</tr>
<tr>
<td>1</td>
<td>100</td>
<td>5</td>
<td>9.7855</td>
<td>9.7845</td>
</tr>
<tr>
<td>1</td>
<td>100</td>
<td>10</td>
<td>9.9817</td>
<td>9.9815</td>
</tr>
<tr>
<td>1</td>
<td>100</td>
<td>50</td>
<td>10.047</td>
<td>10.0471</td>
</tr>
<tr>
<td>1</td>
<td>100</td>
<td>100</td>
<td>10.0492</td>
<td>10.0492</td>
</tr>
<tr>
<td>100</td>
<td>1</td>
<td>0.1</td>
<td>10.0004</td>
<td>10.0004</td>
</tr>
<tr>
<td>100</td>
<td>1</td>
<td>1</td>
<td>10.0212</td>
<td>10.0209</td>
</tr>
<tr>
<td>100</td>
<td>1</td>
<td>5</td>
<td>10.0473</td>
<td>10.0473</td>
</tr>
<tr>
<td>100</td>
<td>1</td>
<td>10</td>
<td>10.0492</td>
<td>10.0492</td>
</tr>
<tr>
<td>100</td>
<td>1</td>
<td>50</td>
<td>10.0498</td>
<td>10.0498</td>
</tr>
<tr>
<td>100</td>
<td>1</td>
<td>100</td>
<td>10.0499</td>
<td>10.0499</td>
</tr>
<tr>
<td>100</td>
<td>100</td>
<td>0.1</td>
<td>10.0371</td>
<td>10.0357</td>
</tr>
<tr>
<td>100</td>
<td>100</td>
<td>1</td>
<td>11.9358</td>
<td>11.9105</td>
</tr>
<tr>
<td>100</td>
<td>100</td>
<td>5</td>
<td>13.9555</td>
<td>13.9548</td>
</tr>
<tr>
<td>100</td>
<td>100</td>
<td>10</td>
<td>14.0937</td>
<td>14.0936</td>
</tr>
<tr>
<td>100</td>
<td>100</td>
<td>50</td>
<td>14.1402</td>
<td>14.1402</td>
</tr>
<tr>
<td>100</td>
<td>100</td>
<td>100</td>
<td>14.1416</td>
<td>14.1416</td>
</tr>
</tbody>
</table>
4 Conclusion

In this work, the equivalent linearization method in conjunction with a weighted averaging is applied in analysis of a nonlinear Duffing – harmonic oscillator. This method is a development of the classical equivalent linearisation method. In the proposed method, the averaging value is calculated in a new way called the weighted averaging value by introducing a weighted coefficient function $h(t)$. Accuracy of the proposed method is verified by comparing the obtained results with the published ones for some specific cases. The proposed method is very simple for applying and gives the results in very high accuracy for both nonlinear systems. The present method will be an effective tool in analyzing nonlinear oscillations.

Acknowledgements

This work is supported by Thai Nguyen University of Technology (TNUT) (no. “T2018-B27”).

Competing Interests

Author has declared that no competing interests exist.

References


© 2019 Hieu–Dang; This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sdiarticle4.com/review-history/52367