Some Geometric Properties of a Generalized Difference Sequence Space Involving Lacunary Sequence

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Author’s contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

In this paper, we define a new generalized difference sequence space \( l(p, θ, Δ^m, s) \) involving lacunary sequence where \( p = (p_r) \) is a bounded sequence of positive real numbers with \( p_r > 1 \) for all \( r \in \mathbb{N} \) and \( s \geq 0 \). Then, we examine the uniform Opial property, k-NUC property and Banach-Saks property of type \( p \) for this space.

Keywords: Difference sequence space; lacunary sequence; Luxemburg norm; uniform Opial property; Fatou property; Banach-Saks of type \( p \).

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1 Introduction

In metric fixed point theory, geometric properties of Banach spaces such as the Opial property, Fatou property and their generalizations play fundamental role. In particular, the Opial property of a Banach space has a great importance in the fixed point theory, differential equation and integral equations. The Opial property is important because Banach spaces with this property have the weak fixed point property.

In [1], Opial has defined the Opial property and proved that $\ell_p(1 < p < \infty)$ satisfies this property but the space $L_p[0, 2\pi](p \neq 2, 1 < p < \infty)$ does not. Franchetti [2] has shown that any infinite dimensional Banach space has an equivalent norm satisfying the Opial property. Later, Prus [3] has introduced and investigated uniform Opial property for Banach spaces. Later on, Petrot and Suantai [4], Mursaleen et al. [5], Mongkolkeha and Kumam [6], Şimşek et al. [7] and many others have studied the uniform Opial property for Cesàro-Orlicz spaces.

Clarkson [8] has introduced the concept of uniform convexity and it is known that uniform convexity implies reflexivity of Banach spaces. Huff [9] has introduced the concept of nearly uniform convexity of Banach spaces. Rolewicz [10] has shown that the Banach space $X$ is reflexive if $X$ has drop property. Later on, Montesinon [11] has extended this result and proved that $X$ has drop property if and only if $X$ is reflexive and has property H. Kutzarova [12] has given a characterization of $k-NUC$ Banach spaces.

In 1930, Banach and Saks [13] has shown that for $1 < p < \infty$, every bounded sequence in $L_p(0, 1)$ has a subsequence whose arithmetic means converge in norm. The class of spaces with the Banach-Saks property is closed under isomorphic copies, subspaces and quotients; it seems unknown whether $X^*$ possesses this property if $X$ does.

By a lacunary sequence $\theta = (k_r)$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by $\theta$ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $k_r - k_{r-1}$ will be denoted by $q_r$. Freedman et al. [14] have defined the space of lacunary strongly convergent sequences. It is well known that there exists very close connection between the space of lacunary strongly convergent sequences and the space of strongly Cesàro summable sequences. Later on, Et [15], Kara and İlkhan [16, 17], Kara et al. [18] and many others have studied different types of sequence spaces using lacunary sequence. Karakaya [19] has introduced a new sequence space involving lacunary sequences connected with Cesàro sequence space and examined some geometric properties of this space equipped with Luxemburg norm. Later on, Karakas et al. [20] and many others constructed different sequence spaces using lacunary sequence and studied various geometric properties for those sequence spaces.

The notion of difference sequence spaces was introduced by Kızmaz [21] and it was generalized by Et and Çolak [22]. Later on, the difference sequence space have been studied by many authors.

2 Definitions and Preliminaries

Throughout this paper, we shall denote $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^+$ as the set of natural numbers, of real numbers and of non-negative real numbers respectively. Let $\ell^0$ be the space of all real sequences $x = (x(k))_{k=1}^\infty$ and $(X, \| \cdot \|)$ be a subspace of $\ell^0$ and a Banach space. Let $S[X]$ and $B(X)$ denotes the unit sphere and closed unit ball respectively.
A sequence \((x_n) \subset X\) is said to be \(\varepsilon\)-separated sequence for some \(\varepsilon > 0\), if separation of sequence \((x_n)\) denoted by \(\text{sep}(x_n) = \inf \{\|x_n - x_m\| : n \neq m\} > \varepsilon\) (Huff [9]).

A Banach space \(X\) is said to have the Opial property ([3]), if for every weakly null sequence \((x_n) \subset X\) and every non-zero \(x \in X\), we have

\[
\liminf_{n \to \infty} \|x_n\| < \liminf_{n \to \infty} \|x_n + x\|. 
\]

A Banach space \(X\) is said to have the uniform Opial property ([3]), if for each \(\varepsilon > 0\), there exists \(\mu > 0\) such that for any weakly null sequence \((x_n)\) in \(S(X)\) and \(x \in X\) with \(\|x\| \geq \varepsilon\), the following inequality holds:

\[
1 + \mu \leq \liminf_{n \to \infty} \|x_n + x\|. 
\]

In any Banach space \(X\), an Opial property is important because it ensures that \(X\) has a weak fixed point property ([23]). Opial in ([1]) has shown that the space \(L_p([0, 2\pi][p \neq 2, 1 < p < \infty])\) does not have this property but the Lebesgue sequence space \(\ell_p([1 < p < \infty])\) has.

Let \((E, \| \cdot \|_E)\) be a real normed linear subspace of \(\ell^p\). \(E\) is said to be a normed sequence lattice ([24]) if it satisfies the following two conditions:

(i) For any \(x \in E\) and \(y \in \ell^p\) such that \(|y[k]| \leq |x[k]|\) for every \(k \in \mathbb{N}\), then \(y \in E\) and \(\|y\|_E \leq \|x\|_E\).

(ii) There exists a sequence \(x = (x(k))_{k=1}^{\infty} \in E\) such that \(x(k) > 0\) for all \(k \in \mathbb{N}\).

A normed sequence lattice \((E, \| \cdot \|_E)\) with complete norm \(\| \cdot \|_E\) is called a Banach sequence lattice (see [24]). In many literatures, a Banach sequence lattice \(E\) is also called a Köthe sequence space ([4],[25]).

A Banach space \(X\) is said to have Banach-Saks property of type \(p(1 < p < \infty)\) if every weakly null sequence \((x_n)\) has a subsequence \((x_{n_k})\) such that for some \(c > 0\),

\[
\left\| \sum_{j=0}^{n} x_{n_k} \right\| < c(n + 1)^{\frac{1}{p}} \quad \text{for all } n \in \mathbb{N}.
\]

A Banach space \(X\) is called uniformly convex (UC) if for each \(\varepsilon > 0\), there is \(\delta > 0\) such that for \(x, y \in S(X)\), the inequality \(\|x - y\| > \varepsilon\) implies that \(\|\frac{1}{2}(x + y)\| < 1 - \delta\). For any \(x \not\in X\), the drop determined by \(x\) is the set \(D(x, B(X)) = \text{conv}(\{x\} \cup B(X))\). A Banach space \(X\) has the drop property (D), if for every closed set \(C\) disjoint with \(B(X)\), there exists an element \(x \in C\) such that \(D(x, B(X)) \cap C = \{x\}\).

A Banach space \(X\) is called nearly uniformly convex (NUC) if for every \(\varepsilon > 0\), there exists \(\delta \in (0, 1)\) such that for every \((x_n) \subseteq B(X)\) with \(\text{sep}(x_n) > \varepsilon\), we have \(\text{conv}(x_n) \cup (1 - \delta)|B(X)| \neq \emptyset\). Huff [9] has proved that every NUC Banach space is reflexive and has property (H).

Kutzarova [12] has given a characterization of \(k\)-nearly uniformly convex Banach spaces. Let \(k \geq 2\) be an integer. A Banach space \(X\) is said to be \(k\)-nearly uniformly convex \((k - \text{NUC})\), if for any \(\varepsilon > 0\), there exists \(\delta > 0\) such that for any sequence \((x_n) \subset B(X)\) with \(\text{sep}(x_n) > \varepsilon\), there are \(n_1, n_2, \ldots, n_k \in \mathbb{N}\) such that...
If $x$ for all $l$.

Now, we define the generalized difference sequence space

$$
\|kx + k^{2} + \cdots + k^{n}\| < 1 - \delta.
$$

It is clear that $k - NUC$ Banach spaces are $NUC$ but the opposite does not hold in general.

A Banach sequence lattice $\mathcal{E}$ is said to have the Fatou property, if for any $x \in \ell^{0}$ and sequence $(x_{n}) \subset \mathcal{E}$ (where $\mathcal{E} = \{ x \in \mathcal{E} : x \geq 0 \}$) satisfying $0 \leq x_{n}(k) \nearrow x(k)$, that is, $x_{n}(k)$ increases to $x(k)$ as $n \to \infty$ for each $k \in \mathbb{N}$ and sup

$$n_{x} \| x \| x < \infty,$

then, $x \in \mathcal{E}$ and $\| x \| \mathcal{E} = \lim_{n \to \infty} \| x_{n} \| x$ ([25]).

For a real vector space $X$, a function $\rho : X \to [0, \infty)$ is called a modular if it satisfies the following conditions:

(i) $\rho(x) = 0$ if and only if $x = 0$.

(ii) $\rho(\alpha x) = \rho(x)$ for all $\alpha \in \mathbb{F}$ with $|\alpha| = 1$.

(iii) $\rho(\alpha x + \beta y) \leq \rho(\alpha x) + \rho(y)$ for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

Further, the modular $\rho$ is called convex if

(iv) $\rho(\alpha x + \beta y) \leq \alpha \rho(x) + \beta \rho(y)$ for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

For any modular $\rho$ on $X$, the space

$$X_{\rho} = \{ x \in X : \rho(\lambda x) < \infty \text{ for some } \lambda > 0 \}$$

is called the modular space.

A sequence $(x_{n})$ in $X_{\rho}$ is called modular convergent to $x \in X_{\rho}$, if there exists a $\lambda > 0$ such that $\rho(\lambda(x_{n} - x)) \to 0$ as $n \to \infty$. If $\rho$ is a convex modular, the functions

$$\| x \|_{\rho} = \inf \left\{ \lambda > 0 : \rho \left( \frac{x}{\lambda} \right) \leq 1 \right\}$$

and

$$\| x \|_{\rho} = \inf \frac{1}{k>0} \left( 1 + \lambda(kx) \right)$$

are two norms on $X_{\rho}$, which are called the Luxemburg norm and the Amemiya norm respectively. The two norms are equivalent([26]).

A modular $\rho$ is said to satisfy the $\delta_{2}$--condition ($\rho \in \delta_{2}$) if for any $\varepsilon > 0$, there exists constants $K \geq 2$ and $\alpha > 0$ such that

$$\rho (2x) \leq K \rho(x) + \varepsilon$$

for all $x \in X_{\rho}$ with $\rho(x) \leq \alpha$.

If $\rho$ satisfies the $\delta_{2}$--condition for all $\alpha > 0$ with $K \geq 2$ dependent on $\alpha$, we say that $\rho$ satisfies the strong $\delta_{2}$--condition ($\rho \in \delta_{2}^{s}$).

Now, we define the generalized difference sequence space $l(p, \theta, \Delta^{m}, s)$ by

$$l[p, \theta, \Delta^{m}, s] = \{ x \in \ell^{0} : \rho_{\Delta}(\lambda x) < \infty \text{ for some } \lambda > 0 \}$$

where $p = (p_{r})$ is a sequence of positive real numbers with $p_{r} \geq 1$ for all $r \in \mathbb{N}$ and $s \geq 0$, equipped with the Luxemburg norm

$$\| x \| = \inf \left\{ \lambda > 0 : \rho_{\Delta}(\lambda x) \leq 1 \right\}$$
We have

\[
\Delta^m x(k) = \sum_{i=1}^{m} (-1)^i \binom{m}{i} x(k + i) - x(k).
\]

If \( p = \{p_r\} \) is bounded, then

\[
l(p, \theta, \Delta^m) = \left\{ x = (x(k)) : \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in \mathbb{N}} k^{-r} |\Delta^m x(k)| \right)^{p_r} < \infty \right\}.
\]

If \( p_r = p \) for all \( r \in \mathbb{N} \), then the space \( l(p, \theta, \Delta^m) \) reduces to \( l_p(\theta, \Delta^m) \) where

\[
l_p(\theta, \Delta^m) = \left\{ x = (x(k)) : \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in \mathbb{N}} k^{-r} |\Delta^m x(k)| \right)^p < \infty \right\}.
\]

The space \( l(p, \theta, \Delta^m) \) includes the following classes as particular cases:

(i) When \( m = 0, \) this space reduces to the space \( l_p(s, \theta) \) studied by Öztürk and Başarır [27].

(ii) When \( s = 0, \) this space reduces to the space \( l(p, \theta, \Delta^m) \) studied by Karakaş et al. [20].

(iii) When \( \theta = (2^r), s = 0 \) and \( m = 0, \) this space reduces to the space \( ces(p) \) studied by Suantai [28].

(iv) When \( \theta = (2^r, s = 0, m = 0) \) and \( p_r = p \) for all \( r \in \mathbb{N}, \) this space reduces to the space \( ces_{p_r} \) studied by Shiue [29].

**Notations:** For any \( x \in \ell^p \) and \( i \in \mathbb{N}, \) we use the following notations throughout the paper:

\[
x_i = (x(1), x(2), \ldots, x(k), 0, 0, \ldots), \text{ called the truncation of } x \text{ at } k,
\]

\[
x_{\leq k} = (x(1), x(2), \ldots, x(k), 0, 0, \ldots), \text{ called the truncation of } x \text{ at } k,
\]

\[
x_{\geq k} = (x(k), x(k+1), x(k+2), \ldots),
\]

\[
x_{\uparrow} = \{x = (x(k)) \in \ell^p : x(k) \neq 0 \text{ for all } k \in I \subseteq \mathbb{N} \text{ and } x(k) = 0 \text{ for all } k \in \mathbb{N}\setminus I, \text{ and } \text{supp} x = \{k \in \mathbb{N} : x(k) \neq 0\}
\]

and \( clA \) denotes the closure of a set \( A. \)

Throughout this paper, we assume that \( p = (p_r) \) is bounded with \( p_r > 1 \) for all \( r \in \mathbb{N}. \)

## 3 Main Results

In this part of the paper, our goal is to show that the space \( l(p, \theta, \Delta^m) \) has uniform Opial property, Fatou property and Banach Saks of type \( p. \)

**Lemma 3.1.** The functional \( \rho_{\Delta^m} \) on \( l(p, \theta, \Delta^m) \) is a convex modular.

**Proof.** We have \( \rho_{\Delta^m} (x) = 0. \)

\[
\iff \sum_{i=1}^{m} |x(i)| + \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in \mathbb{N}} k^{-r} |\Delta^m x(k)| \right)^{p_r} = 0.
\]

\[
\iff \sum_{i=1}^{m} |x(i)| = 0 \text{ and } \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in \mathbb{N}} k^{-r} |\Delta^m x(k)| \right)^{p_r} = 0.
\]
\( \iff x(i) = 0 \text{ for } i = 1, 2, \ldots, m \text{ and } \Delta^m x(k) = 0 \text{ for all } k \in I, r \in \mathbb{N}. \)

\( \iff x = 0. \)

It is obvious that \( \rho_{\Delta^m}(ax) = \rho_{\Delta^m}(x) \) for all scalar \( a \) with \( |a| = 1. \)

If \( x, y \in l(p, \theta, \Delta^m, s) \) and \( \alpha \geq 0, \beta \geq 0 \) with \( \alpha + \beta = 1 \), by the convexity of the function \( t \rightarrow |t|^r \) for every \( r \in \mathbb{N} \) and the linearity property of the operator \( \Delta^m \), we have

\[
\rho_{\Delta^m}(ax + \beta y) = \sum_{i=1}^{m} |ax(i) + \beta y(i)| + \sum_{r=1}^{\infty} \left( \frac{1}{h^r} \sum_{k \in I_r} k^{-r} \| \alpha \Delta^m x(k) + \beta \Delta^m y(k) \| \right)^{pr} \\
\leq \sum_{i=1}^{m} (|ax(i)| + |\beta y(i)|) + \sum_{r=1}^{\infty} \left( \frac{1}{h^r} \sum_{k \in I_r} k^{-r} \| \alpha \Delta^m x(k) \| \right)^{pr} \\
\leq \sum_{i=1}^{m} (|ax(i)| + |\beta y(i)|) + \alpha \sum_{r=1}^{\infty} \left( \frac{1}{h^r} \sum_{k \in I_r} k^{-r} \| \Delta^m x(k) \| \right)^{pr} \\
\quad \quad + \beta \sum_{r=1}^{\infty} \left( \frac{1}{h^r} \sum_{k \in I_r} k^{-r} \| \Delta^m y(k) \| \right)^{pr} \\
= \alpha \rho_{\Delta^m}(x) + \beta \rho_{\Delta^m}(y) \\
\]

Hence \( \rho_{\Delta^m} \) is a convex modular on \( l(p, \theta, \Delta^m, s) \).

**Lemma 3.2.** For \( x \in l(p, \theta, \Delta^m, s) \), the modular \( \rho_{\Delta^m} \) on \( l(p, \theta, \Delta^m, s) \) satisfies the following properties:

1. if \( 0 < \alpha < 1 \), then \( \alpha \rho_{\Delta^m}(x) \leq \rho_{\Delta^m}(x) \) and \( \rho_{\Delta^m}(ax) \leq \alpha \rho_{\Delta^m}(x) \).

2. if \( \alpha > 1 \), then \( \rho_{\Delta^m}(x) \leq \alpha \rho_{\Delta^m}(x) \)

3. if \( \alpha \geq 1 \), then \( \rho_{\Delta^m}(x) \leq \alpha \rho_{\Delta^m}(x) \leq \rho_{\Delta^m}(ax) \).

**Lemma 3.3.** For any \( x \in l(p, \theta, \Delta^m, s) \), we have

1. if \( \| x \| < 1 \), then \( \rho_{\Delta^m}(x) \leq \| x \| \).

2. if \( \| x \| > 1 \), then \( \rho_{\Delta^m}(x) \geq \| x \| \).

3. if \( \| x \| = 1 \) if and only if \( \rho_{\Delta^m}(x) = 1 \).

4. if \( \| x \| < 1 \) if and only if \( \rho_{\Delta^m}(x) < 1 \).

5. if \( \| x \| > 1 \) if and only if \( \rho_{\Delta^m}(x) > 1 \).

6. if \( 0 < \alpha < 1 \) and \( \| x \| > \alpha \), then \( \rho_{\Delta^m}(x) > \alpha \).

7. if \( \alpha > 1 \) and \( \| x \| < \alpha \), then \( \rho_{\Delta^m}(x) < \alpha \).

**Lemma 3.4.** If \( \rho_{\Delta^m} \in \Delta^2_\mathcal{L} \), then for any \( L > 0 \) and \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

\[
|\rho_{\Delta^m}(x + y) - \rho_{\Delta^m}(x)| < \epsilon
\]

where \( x, y \in l(p, \theta, \Delta^m, s) \) with \( \rho_{\Delta^m}(x) \leq L \) and \( \rho_{\Delta^m}(y) \leq \delta \).
Lemma 3.5. (i) If $\rho_{\Delta}^m \in \Delta^s_2$, then for any $x \in l(p, \theta, \Delta^m, s)$, $\| x \| = 1$ if and only if $\rho_{\Delta}^m(x) = 1$.

(ii) If $\rho_{\Delta}^m \in \Delta^s_2$, then for any $(x_n) \in l(p, \theta, \Delta^m, s)$, $\| x_n \| \to 0$ if and only if $\rho_{\Delta}^m(x_n) \to 0$.

Lemma 3.6. If $\rho_{\Delta}^m \in \Delta^s_2$, then for any $\varepsilon \in (0, 1)$, there exists $\delta \in (0, 1)$ such that $\rho_{\Delta}^m(x) \leq 1 - \varepsilon$ implies $\| x \| \leq 1 - \delta$.

Lemma 3.7. If $\rho_{\Delta}^m \in \Delta^s_2$, then for any $\delta = \delta(\varepsilon) > 0$ such that $\| x \| \geq 1 + \delta$ whenever $\rho_{\Delta}^m(x) \geq 1 + \varepsilon$.

Theorem 3.1. The sequence space $l(p, \theta, \Delta^m, s)$ is a complete paranormed space (not necessarily total paranormed) with

$$g_{\Delta}^m(x) = \sum_{i=1}^{m} |x(i)| + \left(\sum_{i=1}^{\infty} \left(\frac{1}{p_i} \sum_{k \in I_i} k^{-1} |\Delta^m x(k)|\right)^{p_i}\right)^{\frac{1}{p}}$$

where $M = \max(1, H = \sup p_i)$ and the sequence space $l_p(\theta, \Delta^m, s)$ is a BK-space normed by

$$\| x \|_{\Delta^m} = \sum_{i=1}^{m} |x(i)| + \left(\sum_{i=1}^{\infty} \left(\frac{1}{p_i} \sum_{k \in I_i} k^{-1} |\Delta^m x(k)|\right)^{p_i}\right)^{\frac{1}{p}}, 1 < p < \infty.$$

It is easy to see that $\| x \| = \| x \|_{\Delta^m}$, that is, the Luxemburg norm on $l(p, \theta, \Delta^m, s)$ can be reduced to a usual norm on $l_p(\theta, \Delta^m, s)$.

Theorem 3.2. The sequence space $l(p, \theta, \Delta^m, s)$ is a Banach space normed by equation (4).

Theorem 3.3. The space $l(p, \theta, \Delta^m, s)$ has Fatou property.

That is, if $x = (x(i))_{i=1}^{\infty} \in \ell^p$, $(x_n) \in l(p, \theta, \Delta^m, s)$, $x_n = (x_n(i))_{i=1}^{\infty}$, $n \in \mathbb{N}$ are such that $0 \leq x_n(i) \not\to x(i)$ as $n \to \infty$ for each $i \in \mathbb{N}$ and $\sup \| x_n \| < \infty$, then $x \in l(p, \theta, \Delta^m)$ and $\| x \| = \lim_{n \to \infty} \| x_n \|$.

Proof. Assume that $x_n \in l(p, \theta, \Delta^m, s)$ for all $n \in \mathbb{N}$, $\sup \| x_n \| < \infty$ and $0 \leq x_n(i) \not\to x(i)$ as $n \to \infty$ for each $i \in \mathbb{N}$.

Let $A = \sup_n \| x_n \|$. We know that $\| x_n \| \leq A < \infty$ for all $n \in \mathbb{N}$. So $0 \leq \frac{x_n(i)}{x_n} \leq \frac{x_n}{\| x_n \|}$ for all $n \in \mathbb{N}$.

Therefore, $\rho_{\Delta}^m\left(\frac{x_n(i)}{x_n}\right) \leq 1$ and since the modular $\rho_{\Delta}^m$ is monotone, we get $\rho_{\Delta}^m\left(\frac{x_n(i)}{x_n}\right) \leq \rho_{\Delta}^m\left(\frac{x_n}{\| x_n \|}\right) \leq 1$.

Then, by the Beppo Levi theorem and the fact that $A^{-1}x_n \to A^{-1}x$ as $n \to \infty$, we get

$$\rho_{\Delta}^m\left(\frac{x}{A}\right) = \lim_{n \to \infty} \rho_{\Delta}^m\left(\frac{x_n}{A}\right) = \sup_n \rho_{\Delta}^m\left(\frac{x_n}{A}\right) \leq 1.$$

\(\Rightarrow\) $\| x \| \leq A$ and $x \in l(p, \theta, \Delta^m, s)$.

Again, since $\sup_n \| x_n \| < \infty$ and $(\| x_n \|)$ is non-decreasing, so we have $\| x_n \| \to A = \sup_n \| x_n \|$ as $n \to \infty$. 

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By the definition of norm $\|x_n\|$, we have

$$\|x_n\| = \inf \left\{ \lambda > 0 : \rho_{\Delta^m} \left( \frac{x_n}{\lambda} \right) \leq 1 \right\}$$

$$= \inf \left\{ \lambda > 0 : \sum_{i=1}^{m} \frac{|x_n(i)|}{\lambda} + \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} k^{-s} |\Delta^m x_n(k)| \right)^{p_r} \leq 1 \right\}$$

$$\leq \inf \left\{ \lambda > 0 : \sum_{i=1}^{m} \frac{|x(i)|}{\lambda} + \sum_{r=1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} k^{-s} |\Delta^m x(k)| \right)^{p_r} \leq 1 \right\}$$

$$= \inf \left\{ \lambda > 0 : \rho_{\Delta^m} \left( \frac{x}{\lambda} \right) \leq 1 \right\}$$

$$= \|x\|$$

$$\Rightarrow \sup_n \|x_n\| \leq \|x\|$$. Hence, we have $\|x\| = \sup_n \|x_n\| = \lim_{n \to \infty} \|x_n\|$. \qed

**Theorem 3.4.** If $\lim \sup \ p_r < \infty$, then the space $l(p, \theta, \Delta^m, s)$ has uniform Opial property.

**Proof.** Let $\varepsilon > 0$ be any arbitrary number and $x \in l(p, \theta, \Delta^m, s)$ with $\|x\| \geq \varepsilon$. Let $(x_n)$ be any weakly null sequence in $S(l(p, \theta, \Delta^m, s)$.

Since, $\lim \sup \ p_r < \infty$, that is, $\rho_{\Delta^m} \in \delta_2$, by Lemma 3.5(ii), for each $\varepsilon > 0$, there is a $\delta \in (0, 1)$ such that for each $x \in l(p, \theta, \Delta^m, s)$ we have $\rho_{\Delta^m}(x) \geq \delta$. Again, since $\rho_{\Delta^m} \in \delta_2$, Lemma 3.4 for any $\varepsilon > 0$, there exists $\delta_1 \in (0, \delta)$ such that

$$|\rho_{\Delta^m}(u + v) - \rho_{\Delta^m}(u)| < \frac{\delta}{4}$$

for any $u, v \in l(p, \theta, \Delta^m, s)$ (5)

whenever $\rho_{\Delta^m}(u) \leq 1$ and $\rho_{\Delta^m}(v) \leq \delta_1$.

Since $\rho_{\Delta^m}(x) < \infty$, so there exists a natural number $r_0 \in \mathbb{N}$ such that

$$\sum_{i=1}^{m} \frac{|x(i)|}{h_i} + \sum_{r=r_0+1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} k^{-s} |\Delta^m x(k)| \right)^{p_r} \leq \frac{\delta_1}{4}$$

(6)

From equation (6), it follows that

$$\delta \leq \sum_{i=1}^{m} \frac{|x(i)|}{h_i} + \sum_{r=1}^{r_0} \left( \frac{1}{h_r} \sum_{k \in I_r} k^{-s} |\Delta^m x(k)| \right)^{p_r} + \sum_{r=r_0+1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} k^{-s} |\Delta^m x(k)| \right)^{p_r}$$

$$\leq \sum_{r=1}^{r_0} \left( \frac{1}{h_r} \sum_{k \in I_r} k^{-s} |\Delta^m x(k)| \right)^{p_r} + \frac{\delta_1}{4}$$

which implies

$$\sum_{r=1}^{r_0} \left( \frac{1}{h_r} \sum_{k \in I_r} k^{-s} |\Delta^m x(k)| \right)^{p_r} \geq \delta - \frac{\delta_1}{4} > \delta - \frac{\delta}{4} = \frac{3\delta}{4}.$$ (7)

By the linearity of the operator $\Delta^m$ and weak convergence implies coordinatewise convergence, that is, $x_n \to 0$.
weakly implies $x_n(i) \to 0$ for each $i \in \mathbb{N}$, so there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we get

$$\sum_{r=1}^{r_0} \left( \frac{1}{h_r} \sum_{k \in I_r} k^{-i}\lvert \Delta^m x_n(k) + \Delta^m x(k) \rvert \right)^{p_r} > \frac{3\delta}{4} \quad (8)$$

Again, using the fact that $x_n \overset{w}{\to} 0$, we can choose $r_0$ such that

$$\rho_{\Delta^m} (x_n | r_0) \to 0 \text{ as } n \to \infty.$$ 

So, there exists a $n_1 > n_0$ such that

$$\rho_{\Delta^m} (x_n | r_0) \leq \delta_1 \text{ for all } n \geq n_1.$$ 

Since, $(x_n) \in S[l(p, \theta, \Delta^m, s), \theta]$, that is, $\|x_n\| = 1$, so by Lemma 3.5(i), we have $\rho_{\Delta^m} (x_n) = 1$. This implies that there exists $r_0$ such that $\rho_{\Delta^m} (x_n|\delta_1) \leq 1$. Now, choose $u = x_n|\delta_1$ and $v = x_n|r_0$. Then, $u, v \in l(p, \theta, \Delta^m, s)$, $\rho_{\Delta^m} (u) \leq 1$ and $\rho_{\Delta^m} (v) \leq \delta_1$. So from equation (5), for all $n \geq n_1$, we have

$$|\rho_{\Delta^m} (x_n|\delta_1) - \rho_{\Delta^m} (x_n|r_0)| \leq \frac{\delta}{4},$$

which implies that $\rho_{\Delta^m} (x_n) - \frac{\delta}{4} < \rho_{\Delta^m} (x_n|\delta_1)$ for all $n \geq n_1$.

That is,

$$\sum_{i=1}^{m} |x_n(i)| + \sum_{r=r_0+1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} k^{-i}\lvert \Delta^m x_n(k) \rvert \right)^{p_r} > 1 - \frac{\delta}{4} \text{ for all } n \geq n_1 \quad (9)$$

Again, since $\rho_{\Delta^m} (x_n|\delta_1) \leq 1$ and $\rho_{\Delta^m} (x_n|\delta_1) \leq \frac{\delta}{4} < \delta_1$, so from equation (5), we have

$$|\rho_{\Delta^m} (x_n|\delta_1) - \rho_{\Delta^m} (x_n|\delta_1)| \leq \frac{\delta}{4},$$

which implies

$$\rho_{\Delta^m} (x_n|\delta_1) > \rho_{\Delta^m} (x_n|\delta_1) - \frac{\delta}{4}, \quad (10)$$

Now, from equations (8), (9), (10) and the linearity property of the operator $\Delta^m$, we have

$$\rho_{\Delta^m} (x_n + x) = \sum_{i=1}^{m} |x_n(i)| + |x(i)| + \sum_{r=1}^{r_0} \left( \frac{1}{h_r} \sum_{k \in I_r} k^{-i}\lvert \Delta^m x_n(k) + \Delta^m x(k) \rvert \right)^{p_r} + \sum_{r=r_0+1}^{\infty} \left( \frac{1}{h_r} \sum_{k \in I_r} k^{-i}\lvert \Delta^m x_n(k) + \Delta^m x(k) \rvert \right)^{p_r} > \sum_{r=1}^{r_0} \left( \frac{1}{h_r} \sum_{k \in I_r} k^{-i}\lvert \Delta^m x_n(k) + \Delta^m x(k) \rvert \right)^{p_r} + \rho_{\Delta^m} (x_n|\delta_1) - \frac{\delta}{4} > \frac{3\delta}{4} + \left( 1 - \frac{\delta}{4} \right) \frac{\delta}{4} = 1 + \frac{\delta}{4}$$

Since, $\rho_{\Delta^m} \in \delta_2$, so by Lemma 3.7, there is a $\mu > 0$ depending only on $\delta$ such that $\|x_n + x\| \geq 1 + \mu$ which
implies that \( \lim \inf_{n \to \infty} \| x_n + x \| \geq 1 + \mu. \)

\[ \text{Theorem 3.5.} \quad \text{The space } l_p(\emptyset, \Delta^m, s) \text{ has the Banach Saks property of type } p. \]

\textbf{Proof.} We will prove the case for \( m = 1 \) and the general case can be followed on the same lines. Let \( \{\xi_n\} \) be a sequence of positive integers for which \( \sum_{n=1}^{\infty} \xi_n \leq \frac{1}{2}. \)

Let \( (x_n) \) be a weakly null sequence in \( B(l_p(\emptyset, \Delta, s)) \).

Set \( x_0 = 0 \) and \( z_1 = x_{n_1} = \Delta x_1. \) Then, there exists \( u_1 \in \mathbb{N} \) such that

\[ \left\| \sum_{i=1}^{u_1} z_1(i)e_i \right\|_\Delta < \xi_1. \]

Since, \( x_n \xrightarrow{w} 0 \) implies that \( x_n \to 0 \) coordinatewise, there exists \( n_2 \in \mathbb{N} \) such that

\[ \left\| \sum_{i=1}^{u_1} x_n(i)e_i \right\|_\Delta < \xi_1 \text{ when } n \geq n_1. \]

Set \( z_2 = x_{n_2} = \Delta x_2. \) Then, there exists \( u_2 > u_1 \) such that

\[ \left\| \sum_{i=1}^{u_2} z_2(i)e_i \right\|_\Delta < \xi_2. \]

Again, using the fact that \( x_n \to 0 \) coordinatewise, there exists \( n_3 > n_2 \) such that

\[ \left\| \sum_{i=1}^{u_2} x_n(i)e_i \right\|_\Delta < \xi_2 \text{ when } n \geq n_3. \]

Continuing this process, we can find two increasing sequences \( \{u_i\} \) and \( \{n_i\} \) such that

\[ \left\| \sum_{i=1}^{u_i} x_n(i)e_i \right\|_\Delta < \xi_j \text{ when } n \geq n_{j+1} \]

and

\[ \left\| \sum_{i=1}^{u_i} z_j(i)e_i \right\|_\Delta < \xi_j \]

where \( z_j = x_{n_j} = \Delta x_j. \) Since, \( \xi_{j-1} + \xi_j < 1, \) it can be seen that \( \| x \|_\Delta < 1 \) and so \( \| x \|_p^\Delta < 1. \) Hence, we
have,
\[
\left\| \sum_{j=1}^{n} z_j \right\|_{\Delta} = \left\| \sum_{j=1}^{n} \left( \sum_{i=1}^{u_j-1} z_j(i)e_i + \sum_{i=u_{j-1}+1}^{u_j} z_j(i)e_i + \sum_{i=u_j+1}^{\infty} z_j(i)e_i \right) \right\|_{\Delta} \\
\leq \left\| \sum_{j=1}^{n} \left( \sum_{i=1}^{u_j-1} z_j(i)e_i \right) \right\|_{\Delta} + \left\| \sum_{j=1}^{n} \left( \sum_{i=1}^{u_j} z_j(i)e_i \right) \right\|_{\Delta} + \left\| \sum_{j=1}^{n} \left( \sum_{i=u_j+1}^{\infty} z_j(i)e_i \right) \right\|_{\Delta} \\
\leq \left\| \sum_{j=1}^{n} \left( \sum_{i=u_{j-1}+1}^{u_j} z_j(i)e_i \right) \right\|_{\Delta} + 2 \sum_{j=1}^{n} \varepsilon_j.
\]

But
\[
\left\| \sum_{j=1}^{n} \left( \sum_{i=u_{j-1}+1}^{u_j} z_j(i)e_i \right) \right\|_{\Delta}^p \leq \sum_{j=1}^{n} \left( \sum_{i=u_{j-1}+1}^{u_j} z_j(i)e_i \right) \right\|_{\Delta}^p \\
= \sum_{j=1}^{n} \sum_{i=u_{j-1}+1}^{u_j} \left( \frac{1}{\|z\|} \sum_{k \in I_i} k^{-s} |z_j(k)| \right)^p \\
\leq \sum_{j=1}^{n} \sum_{i=1}^{\infty} \left( \frac{1}{\|z\|} \sum_{k \in I_i} k^{-s} |z_j(k)| \right)^p \leq n.
\]

Hence we obtain
\[
\left\| \sum_{j=1}^{n} \left( \sum_{i=u_{j-1}+1}^{u_j} z_j(i)e_i \right) \right\|_{\Delta} \leq n^{\frac{1}{p}}.
\]

By using the fact that \(1 \leq n^{\frac{1}{p}}\) for all \(n \in \mathbb{N}\) and \(1 \leq p < \infty\), we have
\[
\left\| \sum_{j=1}^{n} z_j \right\|_{\Delta} \leq n^{\frac{1}{p}} + 1 \leq 2n^{\frac{1}{p}}.
\]

Hence \(l_p(\Theta, \Delta, s)\) has the Banach Saks property of type \(p\).

**Theorem 3.6.** The sequence space \(l_p(\Theta, \Delta, s)\) has \(k\) – NUC property for any integer \(k \geq 2\).

**Proof.** Using the technique of Öztürk and Başarir [27], it can be easily proved that \(l_p(\Theta, \Delta, s)\) has \(k\) – NUC property for any integer \(k \geq 2\).

**Corollary 3.1.** The sequence space \(l_p(\Theta, \Delta, s)\) is NUC and has drop property.

## 4 Conclusions

We have introduced a new generalized difference sequence space \(l_p(\Theta, \Delta^m, s)\) involving lacunary sequence where \(p = (p_r)\) is a bounded sequence of positive real numbers with \(p_r > 1\) for all \(r \in \mathbb{N}\) and \(s \geq 0\). Then, we have studied some geometric properties like uniform Opial property, \(k\)-NUC property and Banach-Saks property of type \(p\) for this space.
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Author has declared that no competing interests exist.

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