Some Fixed Point Results of Non-Newtonian Expansion Mappings

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Authors’ contributions

This work was carried out in collaboration between all authors. Authors MU and BP designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author BP managed the analyses of the study. Author RDD managed the literature searches. All authors read and approved the final manuscript.

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Abstract

We manifest some fixed point and common fixed point results for non-Newtonian expansive maps defined on non-Newtonian metric spaces. The results offered in this article comprise non-Newtonian metric generalizations of some fixed point results in the literature.

Keywords: Non-Newtonian metric space; non-Newtonian expansive mapping; fixed point.

1. INTRODUCTION

The idea of non-Newtonian calculus was firstly acquaint by Grossman and Katz [1]. Later, the non-Newtonian calculus is studied by Bashirov et al. [2], Ozyapici et al. [3], Cakmak and Basar [4] and others.
Newtonian real line. Non

We denote by \( \mathbb{R} \) all the numbers that result by successive

\[ r \mapsto \alpha(r) = e^r = s \]

and

\[ s \mapsto \alpha^{-1}(s) = \ln s = r \]

If \( I(r) = r \) for all \( r \in \mathbb{R} \), then \( I \) is called identity function and we know that inverse of the identity function is itself. If \( \alpha = i \), then \( \alpha \) generates the classical arithmetic and if \( \alpha = \exp \), then \( \alpha \) generates geometrical arithmetic. All concepts of \( \alpha \)-arithmetic have similar properties in classical arithmetic. \( \alpha \)-zero, \( \alpha \)-one and all \( \alpha \)-integers are formed as

\[ \ldots, \alpha(-2), \alpha(-1), \alpha(0), \alpha(1), \alpha(2), \ldots \]

The \( \alpha \)-positive numbers are the numbers \( j \in A \) such that \( 0 < j \) and the \( \alpha \)-negative numbers are those for which \( j < 0 \). The \( \alpha \)-zero, \( 0 \), and the \( \alpha \)-one, \( 1 \), turn out to be \( \alpha(0) \) and \( \alpha(1) \). The \( \alpha \)-integers consist of \( 0 \) and all the numbers that result by successive \( \alpha \)-addition of \( 1 \) and \( 0 \) and by successive \( \alpha \)-subtraction of \( 1 \) and \( 0 \).

We denote by \( \mathbb{R}(N) \) the range of generator \( \alpha \) and write \( \mathbb{R}(N) = \{ \alpha(r) : r \in \mathbb{R} \} \). \( \mathbb{R}(N) \) is called Non-Newtonian real line. Non-Newtonian arithmetic operations on \( \mathbb{R}(N) \) are represented as follows:

\[ \alpha \text{-addition} \quad i + j = \alpha(\alpha^{-1}(i) + \alpha^{-1}(j)) \]
The α-square of a number $i \in A \subset \mathbb{R}(N)$ is denoted by $i \times i = i^{2N}$. For each α-nonnegative number $v$, the symbol $\sqrt[N]{v}$ will be used to denote $v = \alpha \left( \sqrt[N]{v} \right)$ which is the unique α-square is equal to $i$, which means that $v^{2N} = i$. Throughout this paper, $i^{pN}$ denotes the $p$th non-Newtonian exponent. Thus we have

$$i^{2N} = i \times i = \alpha(\alpha^{-1}(i) \times \alpha^{-1}(i)) = \alpha([\alpha^{-1}(i)]^2),$$

$$i^{3N} = i^{2N} \times i = \alpha(\alpha^{-1}(i^{2N}) \times \alpha^{-1}(i))$$

$$= \alpha \left( \alpha^{-1} \left( \alpha(\alpha^{-1}(i) \times \alpha^{-1}(i)) \right) \times \alpha^{-1}(i) \right) = \alpha([\alpha^{-1}(i)]^3),$$

$$\vdots$$

$$i^{pN} = i^{p-1N} \times i = \alpha([\alpha^{-1}(i)]^p)$$

The α-absolute value of a number $i \in A \subset \mathbb{R}(N)$ is defined as $\alpha(|\alpha^{-1}(i)|)$ and is denoted by $|i|_N$. For each number $i \in A \subset \mathbb{R}(N)$, $\sqrt[N]{N^N} = |i|_N = \alpha(|\alpha^{-1}(i)|)$. In this case,

$$|i|_N = \begin{cases} 
i, & \text{if } i > 0 \\
0, & \text{if } i = 0 \\
\overline{0} - i, & \text{if } i < 0 \end{cases}$$

Also $\mathbb{R}^+(N)$ denotes non-Newtonian positive real numbers and $\mathbb{R}^-(N)$ denotes non-Newtonian negative real numbers. α-intervals are represented by

Closed α-interval
$$[i, j] = [i, j]_N = \{ s \in \mathbb{R}(N) : i \leq s \leq j \}$$

= $\{ s \in \mathbb{R}(N) : \alpha^{-1}(i) \leq \alpha^{-1}(s) \leq \alpha^{-1}(j) \}$

Open α-interval
$$\langle i, j \rangle = (i, j)_N = \{ s \in \mathbb{R}(N) : i < s < j \}$$

= $\{ s \in \mathbb{R}(N) : \alpha^{-1}(i) < \alpha^{-1}(s) < \alpha^{-1}(j) \}$

Likewise semi-closed and semi-open α-intervals can be represented. For the set $\mathbb{R}(N)$ of non-Newtonian real numbers, the binary operations (+) addition and (×) multiplication are defined by

$$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

$$(i, j) \mapsto i + j = \alpha(\alpha^{-1}(i) + \alpha^{-1}(j))$$

$$\times : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

$$(i, j) \mapsto i \times j = \alpha(\alpha^{-1}(i) \times \alpha^{-1}(j)).$$
The fundamental properties provided in the classical calculus is provided in non-Newtonian calculus, too.

**Lemma 2.1** (see [4]). \((\mathbb{R}(N),+,*\)) is a topologically complete field.

**Lemma 2.2** (see [4]). \([i \times j]_N = |i|_N \times |j|_N \forall i, j \in \mathbb{R}(N)\).

**Lemma 2.3** (see [4]). \(|i+|_N \leq |i|_N+|j|_N, \forall i, j \in \mathbb{R}(N)\).

The non-Newtonian metric spaces provide an alternative to the metric spaces introduced in [4].

**Definition 2.4** (see [4]). Let \(M\) be a non-empty set and \(d_N: M \times M \to \mathbb{R}^+(N)\) be a function such that for all \(i, j, k \in M\);

- (NNN1) \(d_N(i, j) = 0 \iff i = j\)
- (NNN2) \(d_N(i, j) = d_N(j, i)\)
- (NNN3) \(d_N(i, j) \leq d_N(i, k) + d_N(k, j)\).

Then, the map \(d_N\) is called non-Newtonian metric and the pair \((M, d_N)\) is called non-Newtonian metric space.

**Definition 2.5** (see [4]). Let \(M\) be a vector space on \(\mathbb{R}(N)\). If a function \(\| \|_N: M \to \mathbb{R}^+(N)\) satisfies the following axioms for all \(i, j \in M\) and \(\lambda \in \mathbb{R}(N)\):

- (NNN1) \(\|i\|_N = 0 \iff i = 0\)
- (NNN2) \(\|\lambda \times i\|_N = |\lambda|_N \times \|i\|_N\)
- (NNN3) \(\|i+j\|_N \leq \|i\|_N + \|j\|_N\).

Then it is called a non-Newtonian norm on \(M\) and the pair \((M, \| \|_N)\) is called a non-Newtonian normed space.

**Remark 2.6** (see [4]). Here it is easily seen that every non-Newtonian norm \(\| \|_N\) on \(M\) produces a non-Newtonian metric \(d_N\) on \(M\) given by

\[d_N(i, j) = \|i-j\|_N, \forall i, j \in M\]

**Definition 2.7** (see [4]). (Non-Newtonian convergent sequence) A sequence \(\{j_n\}\) in a non-Newtonian metric space \((M, d_N)\) is said to be non-Newtonian convergent if for every given \(\varepsilon > 0\), there exists an \(n_0 = n_0(\varepsilon) \in \mathbb{N}\) and \(j \in M\) such that \(d_N(j_n, j) < \varepsilon\) for all \(n > n_0\) and is denoted by \(\lim_{n \to +\infty} j_n = j\) or \(j_n \longrightarrow j\) as \(n \to \infty\).

**Definition 2.8** (see [4]). (Non-Newtonian Cauchy sequence) A sequence \(\{j_n\}\) in a non-Newtonian metric space \((M, d_N)\) is said to be non-Newtonian Cauchy if for every given \(\varepsilon > 0\), there exists an \(n_0 = n_0(\varepsilon) \in \mathbb{N}\) such that \(d_N(j_m, j_n) < \varepsilon\) for all \(n, m > n_0\).

**Definition 2.9** (see [4]). (Non-Newtonian complete metric space) The space \(M\) is said to be non-Newtonian complete if every non-Newtonian Cauchy sequence in \(M\) converges.

**Definition 2.10** (see [4]). (Non-Newtonian bounded) Let \((M, d_N)\) be a non-Newtonian metric space. The space \(M\) is said to be non-Newtonian bounded if there is a non-Newtonian constant \(\kappa > 0\) such that \(d_N(i, j) \leq \kappa\) for all \(i, j \in M\). The space \(M\) is said to be non-Newtonian unbounded if it is not non-Newtonian bounded.
Proposition 3.1 (see [4]). Suppose that the non-Newtonian metric $d_N$ on $\mathbb{R}(N)$ is such that $d_N(i, j) = |i - j|_N$ for all $i, j \in \mathbb{R}(N)$, then $(\mathbb{R}(N), d_N)$ is a non-Newtonian metric space.

Lemma 2.12 (see [18]). Let $(M, d_N)$ be a non-Newtonian metric space. Then,

1. A non-Newtonian convergent sequence in $M$ is non-Newtonian bounded and its non-Newtonian limit is unique.
2. A non-Newtonian convergent sequence in $M$ is a non-Newtonian Cauchy sequence in $M$.

From the definition of non-Newtonian Cauchy sequence and Lemma 2.12, we can give the following corollary:

Corollary 2.13 (see [18]) A non-Newtonian Cauchy sequence is non-Newtonian bounded.

Lemma 2.14 (see [18]) Suppose $(M, d_N)$ is a non-Newtonian metric space and $i, j, k \in M$. Then

$$|d_N(i, j) - d_N(j, k)|_N \leq d_N(i, k)$$

Definition 2.15 Let $M$ be a set and $f$ a map from $M$ to $M$. A fixed point of $f$ is a solution of the functional equation $f(j) = j, j \in M$. A point $j \in M$ is called common fixed point of two self-mappings $f$ and $g$ on $M$ if $f(j) = g(j) = j$.

Definition 2.16 (see [18]) Suppose $(M, d_N)$ is a non-Newtonian complete metric space. A mapping $f: M \rightarrow M$ is called non-Newtonian Lipschitzian if there exists a non-Newtonian number $\delta \in \mathbb{R}(N)$ such that

$$d_N(f(i), f(j)) \leq \delta \times d_N(i, j), \forall i, j \in M.$$ 

The mapping $f$ is called non-Newtonian contractive if $\delta < 1$.

Binbasıoglu et al. [18] established following result in non-Newtonian metric space.

Theorem 2.17 Let $f$ be a non-Newtonian contraction mapping on a non-Newtonian complete metric space $M$. Then $f$ has a unique fixed point.

3. Main Results

Now, we give some properties related to non-Newtonian metric spaces and non-Newtonian normed spaces.

Proposition 3.1 The non-Newtonian distance is commutative.

Proof Let $i$ and $j$ be any two non-Newtonian numbers. Then

$$|i - j|_N = a(|a^{-1}(i) - a^{-1}(j)|)$$

$$= a(|a^{-1}(j) - a^{-1}(i)|)$$

$$= |j - i|_N \quad (3.1)$$

This shows that non-Newtonian distance is commutative.

Proposition 3.2 Let $(M, d_N)$ be a non-Newtonian metric space and let $i, j, k, l \in M$. Then

$$|d_N(i, j) - d_N(k, l)|_N \leq d_N(i, k) + d_N(j, l) \quad (3.2)$$
Lemma 3.6

Now, we give a simple but a useful Lemma.

Definition 3.5

Now, we introduce some definitions in non-Newtonian metric spaces. Then

\[ ||i||_N - ||j||_N \leq ||i - j||_N, \forall i, j \in M \]  

(3.5)

Proof

Observe that

\[ ||i||_N^2 = ||i - j + j||_N \leq ||i - j||_N + ||j||_N \]

Therefore \[ ||i||_N - ||j||_N \leq ||i - j||_N \]. Swapping the role of \( i \) and \( j \), we also obtain \[ ||j||_N - ||i||_N \leq ||i - j||_N \]. This implies (3.5).

Now, we introduce some definitions in non-Newtonian metric spaces.

Definition 3.4

Suppose \((M, d_N)\) is a non-Newtonian complete metric space. A mapping \( f: M \rightarrow M \) is called non-Newtonian expansive if there exists a non-Newtonian number \( \delta \geq 1 \) such that

\[ d_N(fx, fy) \geq \delta \times d_N(x, y), \forall x, y \in M. \]  

(3.6)

Definition 3.5

Let \((M, d_N)\) be a non-Newtonian metric space and \( f \) be a self-mapping of \( M \): (NN1) There exist non-Newtonian numbers \( a, b, c \) satisfying \( b \geq 0, c \geq 0 \) and \( a > 1 \) such that

\[ d_N(f(i), f(j)) \geq a \times d_N(i, j) + b \times d_N(i, f(i)) + c \times d_N(j, f(j)) \]  

(3.7)

for each \( i, j \in M \). In this case \( f \) is called non-Newtonian expansive type mapping.

Now, we give a simple but a useful Lemma.

Lemma 3.6

Let \( \{j_n\} \) be a sequence in a non-Newtonian metric space such that

\[ d_N(j_n, j_{n+1}) \leq \delta \times d_N(j_{n-1}, j_n) \]  

(3.8)

where \( \delta < 1 \) and \( n \in \mathbb{N} \). Then \( \{j_n\} \) is a non-Newtonian Cauchy sequence in \( M \).
Since

We now show that exists a point Then by Lemma 3.6, where

Proof: Theorem 3.7 Let \( f: M \to M \) be a surjection and non-Newtonian expansive mapping on a non-Newtonian complete metric space \( M \). Then \( f \) has a unique fixed point.

Proof: Let \( j_0 \in M \) be arbitrary. Since \( f \) is surjection, then there exists \( j_1 \in M \) such that \( j_0 = f(j_1) \). By continuing this process, we get

\[
j_n = f(j_{n+1}), \quad n = 0, 1, 2, \ldots
\]  

(3.11)

In case \( j_n = j_{n+1} \) for some \( n \), then it is clear that \( j_n \) is a fixed point of \( f \). Now assume that \( j_n \neq j_{n-1} \) for all \( n \). Since \( f \) non-Newtonian expansive mapping

\[
d_N(j_{n-1}, j_n) = d_N(f(j_n), f(j_{n+1})) \geq \delta \times d_N(j_{n-1}, j_n)
\]

Consequently

\[
d_N(j_n, j_{n+1}) \geq \left( \frac{1}{\delta} \right) \times d_N(j_{n-1}, j_n) = \kappa \times d_N(j_{n-1}, j_n)
\]  

(3.12)

where \( \kappa = 1/\delta < 1 \).

Then by Lemma 3.6, \( \{j_n\} \) is an NN-Cauchy sequence. Since \( (M, d_N) \) is non-Newtonian complete, there exists a point \( j \) in \( M \) such that \( j_n \to j \). Since \( f \) is surjection on \( M \), there exists \( u \in M \) such that \( j = f(u) \). We now show that \( j \) is a fixed point of the mapping \( f \). It follows from (3.6) and (3.11) that

\[
d_N(j_n, j) = d_N(f(j_{n+1}), f(u)) \geq \delta \times d_N(j_{n+1}, u)
\]

Since \( j_n \to j \), it follows that \( d_N(j_{n+1}, u) \to 0 \) and hence \( j_{n+1} \to u \). By uniqueness of non-Newtonian limit, we have \( j = u \). This shows that \( j \) is a fixed point of \( f \). We conclude the proof by showing that \( j \) is the only fixed point. Suppose that \( k \) is also a fixed point, that is, suppose \( f(k) = k \), then

\[
d_N(j, k) = d_N(f(j), f(k)) \geq \delta \times d_N(j, k)
\]

Since \( \delta > 1 \), this implies that \( d_N(j, k) = 0 \) and hence \( j = k \).
**Theorem 3.8** Let \((M, d_N)\) be a non-Newtonian complete metric space and let \(f\) be a surjective self-mapping of \(M\). If \(f\) satisfies condition \((NN1)\), then \(f\) has a unique fixed point in \(M\).

**Proof.** Using the hypothesis, it can be easily seen that \(f\) is injective. Indeed, if we take \(f(i) = f(j)\), then, using (3.7), we get

\[
0 = d_N(f(i), f(j)) \geq a \times d_N(i, f(i)) + b \times d_N(i, f(i)) + c \times d_N(f(i), f(j))
\]

And so \(d_N(i, j) = 0\); that is, we have \(i = j\), since \(a > 1\).

Let us denote the inverse mapping of \(f\) by \(F\). Let \(j_0 \in M\) and define the sequence \(\{j_n\} \) as follows:

\[
\begin{align*}
j_1 &= F(j_0), \\
j_2 &= F(j_1) = F^2(j_0), \\
j_3 &= F(j_2) = F^3(j_0), \ldots, \\
j_n &= F(j_{n-1}) = F^n(j_0), \\
\end{align*}
\]

(3.13)

Suppose that \(j_n \neq j_{n+1}\) for all \(n\). Using (3.7) and (3.13), we have

\[
d_N(j_{n-1}, j_n) = d_N(f^{-1}(j_{n-1}), f^{-1}(j_n)) \\
\geq a \times d_N(f^{-1}(j_{n-1}), f^{-1}(j_n)) + b \times d_N(f^{-1}(j_{n-1}), f^{-1}(j_n)) \\
+ c \times d_N(f^{-1}(j_{n-1}), f^{-1}(j_n)) \\
\geq a \times d_N(F(j_{n-1}), F(j_n)) + b \times d_N(F(j_{n-1}), F(j_n)) + c \times d_N(F(j_{n-1}), F(j_n)) \\
= (a+c) \times d_N(j_{n-1}, j_n) + b \times d_N(j_{n-1}, j_n) + c \times d_N(j_{n-1}, j_n)
\]

which implies that

\[
(1-b) \times d_N(j_{n-1}, j_n) \geq (a+c) \times d_N(j_{n-1}, j_n)
\]

(3.14)

Clearly, we have \(a+c \neq 0\). Hence, we obtain

\[
d_N(j_n, j_{n+1}) \leq \frac{(1-b)}{(a+c)} \times d_N(j_{n-1}, j_n) = \delta \times d_N(j_{n-1}, j_n)
\]

(3.15)

Where \(\delta = (1-b)/(a+c)\), then we get \(\delta < 1\), since \(a+b+c > 1\). Repeating this process in condition (3.15), we find

\[
d_N(j_n, j_{n+1}) \leq \delta^n \times d_N(j_0, j_1)
\]

and by Lemma 3.6, \(\{j_n\}\) is an NN-Cauchy sequence. Since \((M, d_N)\) is non-Newtonian complete, there exists a point \(j \) in \(M\) such that \(j_n \xrightarrow{N} j\) and therefore

\[
d_N(j_n, j) \xrightarrow{N} 0, \quad d_N(j_{n+1}, j) \xrightarrow{N} 0.
\]

Using the subjectivity of hypothesis, there exists \(u \in M\) such that \(j = f(u)\). From (3.7) and (3.13), we have

\[
d_N(j_n, j) = d_N(f(j_{n+1}), f(u)) \\
\geq a \times d_N(j_{n+1}, p) + b \times d_N(j_{n+1}, f(j_{n+1})) + c \times d_N(u, f(u)) \\
= a \times d_N(j_{n+1}, p) + b \times d_N(j_{n+1}, j) + c \times d_N(u, f(u))
\]

If we take limit for \(n \to \infty\), we obtain

\[
0 \geq (a+c) \times d_N(u, j)
\]
which implies that $d_N(u,j) = 0$; that is, we have $j = u$, since $a+c > 1$. This shows that $j$ is a fixed point of $f$.

Now we show the uniqueness of $j$. Let $k$ be another fixed point of $f$ with $j \neq k$. Using (3.7), we get

$$
d_N(j,k) = d_N(f(j),f(k))
\geq a \times d_N(j,k) + b \times d_N(f(j),f(k)) + c \times d_N(k,k)
\geq a \times d_N(j,k) + b \times d_N(j,j) + c \times d_N(k,k)
= a \times d_N(j,k)
$$

which implies that $j = k$, since $a > 1$. Consequently, $f$ has a unique fixed point $j$.

If we take $b = c$ in condition (NN1), then we obtain the following corollary.

**Corollary 3.9** Let $(M, d_N)$ be a non-Newtonian complete metric space and let $f$ be a surjective self-mapping of $M$. If there exist real numbers $a$, $b$ satisfying $b \geq 0$ and $a > 1$ such that

$$
d_N(f(i),f(j)) \geq a \times d_N(i,j) + b \times \max\{d_N(i,f(i)),d_N(j,f(j))\}
$$

for each $i, j \in M$, then $f$ has a unique fixed point in $M$.

Now, we prove following common fixed point result.

**Theorem 3.10** Let $f, g: M \to M$ be two surjective mappings of a non-Newtonian complete metric space $(M, d_N)$. Suppose that $f$ and $g$ satisfying inequalities

$$
d_N(f(g(j)),g(j)) + \kappa \times d_N(f(g(j)),j) \geq a \times d_N(g(j),j)
$$

$$
d_N(g(f(j)),f(j)) + \kappa \times d_N(g(f(j)),j) \geq b \times d_N(f(j),j)
$$

for $j \in M$ and some non-Newtonian real numbers $a, b$ and $\kappa$ with $a-\kappa > 1+k$ and $b-\kappa > 1+k$. If $f$ or $g$ is non-Newtonian continuous, then $f$ and $g$ have a common fixed point in $M$.

**Proof** Let $j_0$ be an arbitrary point in $M$. Since $f$ is surjective, there exists $j_1 \in M$ such that $j_0 = f(j_1)$. Also, since $g$ is surjective, there exists $j_2 \in M$ such that $j_2 = g(j_1)$. Continuing this process, we construct a sequence $(j_n)$ in $M$ such that $j_{2n} = f(j_{2n+1})$ and $j_{2n+1} = g(j_{2n+2})$ for all $n \in \mathbb{N}$. Now for $n \in \mathbb{N}$, by (3.18) we have

$$
d_N\left(f(g(j_{2n+2})), g(j_{2n+2})\right) + \kappa \times d_N(f(g(j_{2n+2})), j_{2n+2}) \geq a \times d_N(g(j_{2n+2}), j_{2n+2})
$$

Thus

$$
d_N(j_{2n+1}, j_{2n+2}) + \kappa \times d_N(j_{2n+1}, j_{2n+2}) \geq a \times d_N(j_{2n+1}, j_{2n+2})
$$

which implies that

$$
d_N(j_{2n+1}, j_{2n+2}) + \kappa \times [d_N(j_{2n+1}, j_{2n+2}) + d_N(j_{2n+1}, j_{2n+2})] \geq a \times d_N(j_{2n+1}, j_{2n+2})
$$

Hence

$$
d_N(j_{2n+1}, j_{2n+2}) \leq [(1+\kappa)/(a-\kappa)] \times d_N(j_{2n}, j_{2n+1})
$$

(3.20)
On other hand, from (3.19), we have

\[ d_N(g(f(j_{2n+1})), f(j_{2n+1})) + \kappa \times d_N(g(f(j_{2n+1})), f(j_{2n+1})) \geq b \times d_N(f(j_{2n+1}), f(j_{2n+1})) \]

Thus

\[ d_N(j_{2n-1}, j_{2n}) + \kappa \times d_N(j_{2n-1}, j_{2n+1}) \geq b \times d_N(j_{2n}, j_{2n+1}) \]

which implies that

\[ d_N(j_{2n-1}, j_{2n}) + \kappa \times [d_N(j_{2n-1}, j_{2n}) + d_N(j_{2n}, j_{2n+1})] \geq b \times d_N(j_{2n}, j_{2n+1}) \]

Hence

\[ d_N(j_{2n}, j_{2n+1}) \leq \left[ \frac{(1+\kappa)}{(b-\kappa)} \right] \times d_N(j_{2n-1}, j_{2n}) \quad (3.21) \]

Let \( \delta = \text{maj}\left[\left(\frac{1+\kappa}{a-\kappa}\right), \left(\frac{1+\kappa}{b-\kappa}\right)\right] < 1 \)

Then by combining (3.20) and (3.21), we have

\[ d_N(j_{n+1}, j_n) \leq \delta \times d_N(j_{n-1}, j_n) \quad (3.22) \]

where \( \delta \in (0,1) \), \( \forall \ n \in \mathbb{N} \). Then by Lemma 3.6, the sequence \( (j_n) \) is an NN-Cauchy sequence. Since \( (M, d_N) \) is non-Newtonian complete, there exists a point \( j \) in \( M \) such that \( j_n \xrightarrow{N} j \). Therefore \( j_{2n+1} \xrightarrow{N} j \) and \( j_{2n+2} \xrightarrow{N} j \) as \( n \to +\infty \). Without loss of generality, we may assume that \( f \) is continuous, then \( f(j_{2n+1}) \xrightarrow{N} f(j) \) as \( n \to +\infty \). But \( f(j_{2n+1}) = j_{2n} \xrightarrow{N} j \) as \( n \to +\infty \). Thus, we have \( f(j) = j \). Since \( g \) is surjection on \( M \), there exists \( u \in M \) such that \( j = g(u) \). We now show that \( j \) is a common fixed point of the mapping \( f \) and \( g \). It follows from (3.18) that

\[ d_N(f(g(u)), g(u)) + \kappa \times d_N(f(g(u)), u) \geq a \times d_N(g(u), u) \]
\[ \Rightarrow \quad 0 + \kappa \times d_N(j, u) \geq a \times d_N(j, u) \]
\[ \Rightarrow \quad 0 \geq (a-\kappa) \times d_N(j, u) \]

Since \( a-\kappa > 1+\kappa \), we conclude that \( d_N(j, u) = 0 \) and consequently \( j = u \). Hence \( f(j) = g(j) = j \). Therefore \( j \) is a common fixed point of \( f \) and \( g \).

By taking \( f = g \) in Corollary 3.9 we have the following Corollary.

**Corollary 3.10** Let \( f: M \to M \) be two surjective mappings of a non-Newtonian complete metric space \( (M, d_N) \). Suppose that \( f \) satisfying inequality

\[ d_N(f^2(j), f(j)) + \kappa \times d_N(f^2(j), j) \geq a \times d_N(f(j), j) \quad (3.24) \]

for \( j \in M \) and some nonnegative real numbers \( a, b \) and \( k \) with \( a-\kappa > 1+\kappa \). If \( f \) is continuous, then \( f \) has a fixed point in \( M \).

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Competing Interests

Authors have declared that no competing interests exist.

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