Existence of Nonoscillation Solutions of Higher-Order Nonlinear Neutral Differential Equations

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Authors’ contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this paper, we consider the following higher-order nonlinear neutral differential equations:

$$\frac{d^n}{dt^n}\left[x(t) + cx(t - \tau)\right] + (-1)^{n+1}\left[P(t)f_1(x(t - \sigma)) - Q(t)f_2(x(t - \delta))\right] = 0, \quad t \geq t_0$$

where $\tau, \sigma, \delta \in R^+$, $c \in R$, $c \neq \pm 1$, and $P(t), Q(t) \in C([t_0, \infty), R^+)$, $f_1(u) \in C(R, R)$, $f_2(u) \geq 0$. We obtain the results which are some sufficient conditions for existence of nonoscillation solutions, special case of the equation has also been studied.

Keywords: Higher-order; differential equation; nonoscillation solutions; existence.

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1 Introduction

In this paper, we shall consider existence of nonoscillation solution of higher-order nonlinear neutral differential equations

\[
\frac{d^n}{dt^n}[x(t) + cx(t - \tau)] + (-1)^{n+1}[P(t)f_1(x(t - \sigma)) - Q(t)f_2(x(t - \delta))] = 0, \quad t \geq t_0
\]  

(1.1)

where \(\tau, \sigma, \delta \in R^+, c \in R, c \neq \pm 1\), and \(P(t), Q(t) \in C([t_0, \infty), R^+), R^+ = (0, +\infty)\). \(f_i(u) \in C(R, R), u f_i(u) > 0\). If \(u > 0\), then \(\exists N_i, \text{ st. } 0 < N_i \leq u, \mid f_i(u) - f_i(v) \mid \leq L_i\mid u - v \mid, i = 1, 2\).

Let \(\mu = \{\tau, \sigma, \delta\}\). By a solution of equation (1.1), we mean a continuously function \(x(t)\) such that equation (1.1) is satisfied for \(t \geq t_0\). By a nonoscillation solution of equation (1.1), we mean a nonoscillation solution of equation (1.1). If \(x(t) + cx(t - \tau)\) is continuously differentiable on \([t_1, \infty)\) and such that equation (1.1) is satisfied for \(t \geq t_1\).

Recently, more and more people are interested in nonoscillatory criteria of differential equations, we refer the reader to [1 – 11], the differential equation in [1]

\[
\frac{d^n}{dt^n}[x(t) + cx(t - \tau)] + (-1)^{n+1}[P(t)x(t - \sigma) - Q(t)x(t - \delta)] = 0, \quad t \geq t_0
\]

studied nonoscillation solution for a family of higher-order linear neutral differential equations with positive and negative coefficients. Our principal goal in this paper is to derive existence of nonoscillation solutions for nonlinear equation (1.1).

2 Existence Theorems

**Theorem 1.** Assume that \(0 < c < 1\) and\n
\[
\int_{t_0}^{\infty} s^{n-1} P(s) ds < \infty, \quad \int_{t_0}^{\infty} s^{n-1} Q(s) ds < \infty.
\]  

(2.1)

Further, assume that there exists a constant \(\alpha > \frac{1}{1+c}\) and a sufficiently large \(t_1 \geq t_0\) such that

\[
P(t) \geq \alpha Q(t), \quad \text{for } t \geq t_1
\]  

(2.2)

Then (1.1) has a bounded nonoscillatory solution.

Proof. By (2.1) and (2.2), there exists a \(t_1\) sufficiently large such that

\[
c + \frac{1}{(n-1)!} \int_{t_1}^{\infty} (s - t)^{n-1}(L_1 P(s) + L_2 Q(s)) ds \leq \theta_1 < 1, \quad \text{for } t \geq t_1
\]  

(2.3)

where \(\theta_1\) is a constant, and

\[
0 \leq \frac{1}{(n-1)!} \int_{t_1}^{\infty} (s - t)^{n-1}(\alpha M P(s) - L_2 Q(s)) ds \leq c - 1 - \alpha M, \quad \text{for } t \geq t_1
\]  

(2.4)

\[
0 \leq \frac{1}{(n-1)!} \int_{t_1}^{\infty} (s - t)^{n-1} Q(s) ds \leq \frac{1 - c - \alpha M - c M}{\alpha M}, \quad \text{for } t \geq t_1
\]  

(2.5)

hold, where \(M\) is positive constant such that

\[
\frac{1 - c}{\alpha} \leq M \leq \frac{1 - c}{c(1 + \alpha)}
\]  

(2.6)

holds.

Let \(X\) be the set of all continuous and bounded functions on \([t_0, \infty)\) with the norm \(\| x \| = \text{sup}_{t \geq t_0}|x(t)|\); we define a closed bounded subset \(\Omega\) of \(X\) as follows:

\[
\Omega = \{x \in X : c M \leq x(t) \leq \alpha M, t \geq t_0\}
\]

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Define an operator $S : \Omega \to X$ as follows:

$$Sx(t) = \begin{cases} 1 - c - cx(t - \tau) + \frac{1}{(n-1)!} \int_0^t (s - t)^{n-1} (P(s)f_1(x(s - \delta)) - Q(s)f_2(x(s - \sigma)))ds, & t \geq t_1, \\
1 - c - cx(t - \tau) + \frac{1}{(n-1)!} \int_t^\infty (s - t)^{n-1} (P(s)f_1(x(s - \delta)) - Q(s)f_2(x(s - \sigma)))ds, & t_0 \leq t \leq t_1. \end{cases}$$

We shall show that $S\Omega \subset \Omega$. In fact, for every $x \in \Omega$, and $t \geq t_1$, using (2.4) and (2.6) we get

$$Sx(t) = 1 - c - cx(t - \tau) + \frac{1}{(n-1)!} \int_0^t (s - t)^{n-1} (P(s)f_1(x(s - \delta)) - Q(s)f_2(x(s - \sigma)))ds$$

$$\leq 1 - c + \frac{1}{(n-1)!} \int_t^\infty (s - t)^{n-1} (\alpha MP(s) - L_2 Q(s))ds$$

$$\leq \alpha M$$

Furthermore, in view of (2.5) and (2.6) we have

$$Sx(t) = 1 - c - cx(t - \tau) + \frac{1}{(n-1)!} \int_t^\infty (s - t)^{n-1} (P(s)f_1(x(s - \delta)) - Q(s)f_2(x(s - \sigma)))ds$$

$$\geq 1 - c - \alpha M - \frac{M \alpha}{(n-1)!} \int_0^t (s - t)^{n-1} Q(s)ds$$

$$\geq cM$$

Thus, we proved that $S\Omega \subset \Omega$.

Now we shall show that operator $S$ is a contraction operator on $\Omega$.

In fact, for $x, y \in \Omega$ and $t > t_1$, we have

$$|Sx(t) - Sy(t)| \leq c|x(t - \tau) - y(t - \tau)| + \frac{1}{(n-1)!} \int_t^\infty (s - t)^{n-1} P(s)|f_1(x(s - \sigma)) - f_1(y(s - \sigma))|ds$$

$$+ \frac{1}{(n-1)!} \int_t^\infty (s - t)^{n-1} Q(s)|f_2(x(s - \delta)) - f_2(y(s - \delta))|ds$$

$$\leq [c + \frac{1}{(n-1)!} \int_t^\infty (s - t)^{n-1} (L_1 P(s) + L_2 Q(s))ds] \| x - y \|$$

$$\leq \theta_1 \| x - y \|$$

This implies that

$$\| Sx - Sy \| \leq \theta_1 \| x - y \|$$

where in view of (2.3), $\theta_1 < 1$, which proves that $S$ is a contraction operator on $\Omega$. Therefore $S$ has a unique fixed point $x$ in $\Omega$, which is obviously a bounded positive solution of equation (1.1). This completes the proof of Theorem 1.

**Theorem 2.** Assume that $1 < c < +\infty$ and that (2.1) holds. Further, assume that there exists a constant $\gamma > \frac{c}{c-1}$ and a sufficiently large $t_1 \geq t_0$ such that

$$P(t) \geq \gamma Q(t), \quad \text{for } t \geq t_1$$

Then (1.1) has a bounded nonoscillatory solution.

Proof. By (2.1) and (2.7), there exists a $t_1$, sufficiently large such that

$$\frac{1}{c}[1 + \frac{1}{(n-1)!} \int_{t+\tau}^\infty (s - t - \tau)^{n-1} (L_1 P(s) + L_2 Q(s))ds] \leq \theta_2 < 1, \quad \text{for } t \geq t_1$$

where $\theta_2$ is a constant, and

$$0 \geq \frac{1}{(n-1)!} \int_{t+\tau}^\infty (s - t - \tau)^{n-1} (\gamma M_1 P(s) - L_2 Q(s))ds \leq 1 - c + c\gamma M_1, \quad \text{for } t \geq t_1$$
Define an operator holds. Let \( X \) holds, where \( M \) is positive constant such that
\[
\frac{c-1}{\gamma c} < M < \frac{c-1}{1+\gamma} \tag{2.11}
\]
holds. Let \( X \) be the set of all continuous and bounded functions on \([t_0, \infty)\) with the norm \( \| x \| = \sup_{t \geq t_0} |x(t)| \), we define a closed bounded subset \( \Omega \) of \( X \) as follows
\[
\Omega = \left\{ x \in X : \frac{M_1}{c} \leq x(t) \leq \gamma M_1, t \geq t_0 \right\}
\]
Define an operator \( S : \Omega \to X \) as follows
\[
S(x) = \begin{cases} 
1 - \frac{t}{c} - \frac{1}{c(n-1)!} \int_{t+\tau}^{\infty} (s-t-\tau)^n (P(s)f_1(x(s-\delta)) - Q(s)f_2(x(s-\sigma)))ds & t \geq t_1, \\
\frac{M_1}{c} x(t) & t < t_1.
\end{cases}
\]
We shall show that \( \Omega \subset \Omega \). In fact, for every \( x \in \Omega \), and \( t \geq t_1 \), using (2.9) and (2.11) we get
\[
S(x) = 1 - \frac{t}{c} - \frac{1}{c(n-1)!} \int_{t+\tau}^{\infty} (s-t-\tau)^n (P(s)f_1(x(s-\delta)) - Q(s)f_2(x(s-\sigma)))ds \\
\leq 1 - \frac{t}{c} + \frac{1}{c(n-1)!} \int_{t+\tau}^{\infty} (s-t-\tau)^n \gamma M_1 P(s) - L_2 Q(s)ds \\
\leq \gamma M_1
\]
Furthermore, in view of (2.10) and (2.11) we have
\[
S(x) = 1 - \frac{t}{c} - \frac{1}{c(n-1)!} \int_{t+\tau}^{\infty} (s-t-\tau)^n \gamma M_1 P(s) - L_2 Q(s)ds \\
\geq \gamma M_1 \\
\geq \frac{M_1}{c}
\]
Thus, we proved that \( \Omega \subset \Omega \). Now we shall show that operator \( S \) is a contraction operator on \( \Omega \). In fact, for \( x, y \in \Omega \) and \( t > t_1 \), we have
\[
|S(x) - S(y)| \leq \frac{1}{c} |x(t) - y(t) + \gamma (t+\tau)| + \frac{1}{c(n-1)!} \int_{t+\tau}^{\infty} (s-t-\tau)^n p(s)(f_1(x(s-\sigma)) - f_1(y(s-\sigma)))ds \\
+ \frac{1}{c(n-1)!} \int_{t+\tau}^{\infty} (s-t-\tau)^n Q(s)f_2(x(s-\delta)) - f_2(y(s-\delta)))ds \\
\leq \frac{1}{c} + \frac{1}{c(n-1)!} \int_{t+\tau}^{\infty} (s-t-\tau)^n (L_1 p(s) + L_2 Q(s))ds \| x - y \| \\
\leq \theta_2 \| x - y \|
\]
This implies that
\[
\| Sx - Sy \| \leq \theta_2 \| x - y \|
\]
where in view of (2.8), \( \theta_2 < 1 \), which proves that \( S \) is a contraction operator on \( \Omega \). Therefore \( S \) has a unique fixed point \( x \) in \( \Omega \), which is obviously a bounded positive solution of equation (1.1). This completes the proof of Theorem 2.

**Theorem 3.** Assume that \(-1 < c < 0\) and that (2.1) holds. Further, assume that there exists a constant \( \beta > 1 \) and a sufficiently large \( t_1 \geq t_0 \) such that
\[
P(t) \geq \beta Q(t), \quad \text{for } t \geq t_1 \tag{2.12}
\]
Then (1.1) has a bounded nonoscillatory solution.

Proof. By (2.1) and (2.12), there exists a $t_1$ sufficiently large such that

$$-c + \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1}(L_1p(s) + L_2Q(s))dsdu \leq \theta_1 < 1, \text{ for } t \geq t_1$$

(2.13)

where $\theta_1$ is a constant, and

$$0 \leq \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1}(\beta M_2 P(s) - L_2Q(s))dsdu \leq (c+1)(\beta M_2 - 1), \text{ for } t \geq t_1$$

(2.14)

hold, and

$$\frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1}Q(s)ds < \frac{(1+c)(1-M_2)}{\beta M_2}$$

(2.15)

where $M_2$ is positive constant such that

$$\frac{1}{\beta} < M_2 < 1$$

(2.16)

holds. Let $X$ be the set of all continuous and bounded functions on $[t_0, \infty)$ with the norm $\|x\| = \sup_{t \geq t_0}|x(t)|$, we define a closed bounded subset $\Omega$ of $X$ as follows

$$\Omega = \{x \in X : M_2 \leq x(t) \leq \beta M_2, t \geq t_0\}$$

Define an operator $S : \Omega \to X$ as follows

$$Sx(t) = \begin{cases} 1 + c - cx(t-\tau) + \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1}(P(s)f_1(x(s-\delta)) - Q(s)f_2(x(s-\sigma)))ds & t \geq t_1, \\ Sx(t_1) & t_0 \leq t \leq t_1. \end{cases}$$

We shall show that $S\Omega \subset \Omega$. In fact, for every $x \in \Omega$, and $t \geq t_1$, using (2.12) and (2.14) we get

$$Sx(t) = 1 + c - cx(t-\tau) + \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1}(P(s)f_1(x(s-\delta)) - Q(s)f_2(x(s-\sigma)))ds$$

$$\leq 1 + c - c\beta M_2 + \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1}(\beta M_2 P(s) - L_2Q(s))ds$$

$$\leq 1 + c - c\beta M_2 + (c+1)(\beta M_2 - 1)$$

$$= \beta M_2$$

Furthermore, in view of (2.15) we have

$$Sx(t) = 1 + c - cx(t-\tau) + \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1}(P(s)f_1(x(s-\delta)) - Q(s)f_2(x(s-\sigma)))ds$$

$$\geq 1 + c - c\beta M_2 - \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1}\beta M_2 Q(s)ds$$

$$\geq 1 + c - c\beta M_2 - (1+c)(1-M_2)$$

$$= \beta M_2$$

Thus, we proved that $S\Omega \subset \Omega$. Now we shall show that operator $S$ is a contraction operator on $\Omega$. In fact, for $x, y \in \Omega$ and $t > t_1$, we have

$$|Sx(t) - Sy(t)| \leq c|z(t-\tau) - y(t-\tau)| + \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1}p(s)|f_1(x(s-\sigma)) - f_1(y(s-\sigma))|ds$$

$$+ \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1}Q(s)|f_2(x(s-\delta)) - f_2(y(s-\delta))|ds$$

$$\leq c + \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1}(L_1p(s) + L_2Q(s))ds \|x - y\|$$

$$\leq \theta_3 \|x - y\|$$
This implies that
\[ \| Sx - Sy \| \leq \theta_3 \| x - y \| \]
where in view of (2.13), \( \theta_3 < 1 \), which proves that \( S \) is a contraction operator on \( \Omega \). Therefore \( S \) has a unique fixed point \( x \) in \( \Omega \), which is obviously a bounded positive solution of equation (1.1). This completes the proof of Theorem 3.

**Theorem 4.** Assume that \(-\infty < c < -1\) and that (2.1) holds. Further, assume that there exists a constant \( h > 1 \) and a sufficiently large \( t_1 \geq t_0 \) such that
\[ P(t) \geq hQ(t), \quad \text{for } t \geq t_1 \tag{2.17} \]
Then (1.1) has a bounded nonoscillatory solution.

**Proof:** The proof is similar to Theorem 2, we omitted.

By Theorems 1-4, we have the following result

**Corollary 1.** Assume that \( c \in \mathbb{R}; c \neq \pm 1 \) and
\[ \int_{t_0}^{\infty} s^{n-1} P(s) ds < \infty. \]
then the neutral differential equation
\[ \frac{d^n}{dt^n}[x(t) + cx(t - \tau)] + (-1)^{n+1} [P(t)f_1(x(t - \sigma))] = 0, \quad t \geq t_0 \tag{2.18} \]
has a bounded nonoscillatory solution.

### 3 Conclusion

In this paper, we have introduced existence of nonoscillatory solutions of differential equations of (1.1), the obtained results are easily applicable. If \( c = 1 \) or \( c = -1 \), we can study existence of nonoscillatory solutions of differential equations of (1.1) in the future work.

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### Competing Interests

Authors have declared that no competing interests exist.

### References


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