\((\phi, \psi)\)-Contraction Condition for Multivalued Mappings in Complete Modular Metric Spaces

Duran Turkoglu\(^1\) and Emine Kilinc\(^2*\)

\(^1\)Department of Mathematics, Faculty of Science, Gazi University Ankara, Turkey.
\(^2\)Department of Mathematics, Institute of Natural and Applied Science, Gazi University Ankara, Turkey.

Authors’ contributions
This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information
DOI: 10.9734/ARJOM/2020/v16i1130242

Editor(s):
(1) Dr. Ruben Dario Ortiz Ortiz, Universidad Michoacana de San Nicolas de Hidalgo, Mexico.
(2) G. Subathra, Vels University, India.
(2) Abdelbasset Felhi, Carthage University, Tunisia.

Reviewers:
(1) G. Subathra, Vels University, India.
(2) Abdelbasset Felhi, Carthage University, Tunisia.

Complete Peer review History: http://www.sdiarticle4.com/review-history/64013

Received: 06 October 2020
Accepted: 13 December 2020
Published: 22 December 2020

Abstract
In this paper we investigated \((\phi, \psi)\)-contraction condition for multivalued type mappings in complete modular metric spaces. Our results are more general than metric versions of these type mappings.

Keywords: Modular metric spaces; fixed point; Hausdorff metric.

2010 Mathematics Subject Classification: 46A80, 47H10, 54E35.

1 Introduction
Let \((X, d)\) be a metric space. A mapping \(T : X \to X\) is a contraction if
\[
d(T(x), T(y)) \leq kd(x, y),
\]

\(^*\)Corresponding author: E-mail: eklncln07@gmail.com;
for all $x, y \in X$, where $k < 1$. The Banach Contraction Mapping Principle appeared in explicit form in Banach’s thesis in 1922 [1]. Because of the simplicity and usefulness of this theory, it has become a very popular tool in solving existence problems in many fields of mathematical analysis. So a number of authors have extended and generalized Banach’s Contraction Principle in many directions. Some of these authors such as Rakotch [2], Boyd and Wong [3], Rhoades [4] investigated weaker contractive conditions using a control function $\alpha : [0, \infty) \to [0, 1)$ in place of the contraction constant $k \in (0, 1)$. After Rhoades, several number of results appeared in fixed point theory. These results could be seen in the papers of Zang and Song [5], Doric [6], Hosseini [7] etc. In 1969 Nadler [8] gave fixed point results for multivalued mappings in metric spaces.

The notion of modular space was given by Nakano [9] and it was intensively developed by Musielak and Orlicz, Koshi and Shimogaki, Yamamuro (see [10, 11, 12]) and others. A lot of mathematicians have been interested in fixed point theory in modular spaces. In 2006, Chistyakov introduced the notion of metric modules inspired by the classical linear modules [13] and in 2008 he gave concept of modular metric spaces generated by $F$-modulars [14]. In 2010 Chistyakov gave the notion of modular metric spaces and properties of these spaces [15] and he gave a fixed point theorem for these spaces in 2011 [16]. After Chistyakov’s results, a lot of authors interested fixed point results in modular metric spaces [17, 18, 16, 19, 20, 21, 22]. Kilıç and Alaca [23] defined $(\varepsilon, k)$–uniformly locally contractive mappings and $\eta$-chainable concept and proved a fixed point theorem for these concepts in a complete modular metric spaces. Kilıç and Alaca [24] proved two main fixed point theorems for commuting mappings in modular metric spaces. In 2014 Khamis and Abdou investigated Hausdorff metric modular in modular metric spaces and proved fixed point theorem for these concepts in a complete modular metric spaces. In 2010 Khamis and Abdou investigated Hausdorff metric modular in modular metric spaces and proved fixed point theorem for these concepts in a complete modular metric spaces. In 2014 Khamis and Abdou investigated Hausdorff metric modular in modular metric spaces and proved fixed point theorem for these concepts in a complete modular metric spaces. In 2014 Khamis and Abdou investigated Hausdorff metric modular in modular metric spaces and proved fixed point theorem for these concepts in a complete modular metric spaces. In 2014 Khamis and Abdou investigated Hausdorff metric modular in modular metric spaces and proved fixed point theorem for these concepts in a complete modular metric spaces.

In this paper we investigated fixed point results for multivalued mappings which satisfies $(\varphi, \psi)$–contraction condition in complete modular metric spaces.

**1.1 Preliminaries**

In this section, we will give some basic concepts and definitions about modular metric spaces which are useful for our results.

**Definition 2.1** ([15], Definition 2.1) Let $X$ be a nonempty set, a function $w : (0, \infty) \times X \times X \to [0, \infty]$ is said to be a metric modular on $X$ if satisfying, for all $x, y, z \in X$ the following condition holds:

1. $w_\lambda (x, y) = 0$ for all $\lambda > 0 \Leftrightarrow x = y$;
2. $w_\lambda (x, y) = w_\lambda (y, x)$ for all $\lambda > 0$;
3. $w_{\lambda + \mu} (x, y) \leq w_\lambda (x, z) + w_\mu (z, y)$ for all $\lambda, \mu > 0$.

If instead of (i), we have only the condition

1. $w_\lambda (x, x) = 0$ for all $\lambda > 0$, then $w$ is said to be a (metric) pseudomodular on $X$.

The main property of a metric modular [15] $w$ on a set $X$ is the following: given $x, y \in X$, the function $0 < \lambda \to w_\lambda (x, y) \in [0, \infty]$ is nonincreasing on $(0, \infty)$. In fact, if $0 < \mu < \lambda$, then (iii), (i) and (ii) imply

$$w_\lambda (x, y) \leq w_{\lambda - \mu} (x, x) + w_\mu (x, y) = w_\mu (x, y).$$

It follows that at each point $\lambda > 0$ the right limit $w_{\lambda + 0} (x, y) = \lim_{\varepsilon \to +0} w_{\lambda + \varepsilon} (x, y)$ and the left limit $w_{\lambda - 0} (x, y) = \lim_{\varepsilon \to -0} w_{\lambda + \varepsilon} (x, y)$ exist in $[0, \infty]$ and the following two inequalities hold:

$$w_{\lambda + 0} (x, y) \leq w_\lambda (x, y) \leq w_{\lambda - 0} (x, y).$$
Theorem 2.1 [21] Let $X_w$ be a complete modular metric space and $T$ a contraction on $X_w$. Then, the sequence $(T^nx)_{n \in \mathbb{N}}$ converges to the unique fixed point of $T$ in $X_w$ for any initial $x \in X_w$.

Now we give some definitions, which are useful for our main results.

Definition 2.2 Let $X_w$ be a modular metric space. Then following definitions exist:

1. The sequence $(x_n)_{n \in \mathbb{N}}$ in $X_w$ is said to be convergent to $x \in X_w$ if $w_1(x_n, x) \to 0$, as $n \to \infty$.
2. The sequence $(x_n)_{n \in \mathbb{N}}$ in $X_w$ is said to be Cauchy if $w_1(x_m, x_n) \to 0$, as $m, n \to \infty$.
3. A subset $C$ of $X_w$ is said to be closed if the limit of a convergent sequence of $C$ always belong to $C$.
4. A subset $C$ of $X_w$ is said to be complete if any Cauchy sequence in $C$ is a convergent sequence and its limit is in $C$.
5. A subset $C$ of $X_w$ is said to be $w$-bounded if

$$\delta_w(C) = \sup \{w_1(x, y); x, y \in C\} < \infty.$$  

6. $w$ is said to satisfy the Fatou property if and only if for any sequence $(x_n)_{n \in \mathbb{N}}$ in $X_w$ $w$-convergent to $x$, we have

$$w_1(x, y) \leq \liminf_{n \to \infty} w_1(x_n, y),$$  

for any $y \in X_w$.

Now we will give some basic properties and notions of multivalued mappings in modular metric spaces which was given in [25] For a subset $M$ of modular metric space $X_w$ set

(i) $\mathcal{CB}(M) = \{ C : C \text{ is nonempty } w - \text{closed and } w - \text{bounded subset of } M \}$;

(ii) $K(M) = \{ C : C \text{ is nonempty } w - \text{compact subset of } M \}$;

(iii) the Haussdorf modular metric is defined on $\mathcal{CB}(M)$ by

$$H_w(A, B) = \max \left\{ \sup_{x \in A} w_1(x, B), \sup_{y \in B} w_1(y, A) \right\},$$

where $w_1(x, B) = \inf_{y \in B} w_1(x, y)$.

Definition 2.3 Let $X_w$ be a complete modular metric space and $M$ be a nonempty subset of $X_w$. A mapping $T : M \to \mathcal{CB}(M)$ is called a multivalued Lipschitzian mapping, if there exists a constant $k > 0$ such that

$$H_w(Tx, Ty) \leq k w_1(x, y),$$

for any $x, y \in M$.

A point $x \in M$ is called fixed point of $T$ whenever $x \in Tx$. The set of fixed points of $T$ will be denoted by Fix($T$).

It was shown in [25] that Definition 2.3 is more general than Theorem 2.1. 

Definition 2.4 A function $\psi : [0, \infty] \to [0, \infty)$ is called an altering distance function if it satisfies the following conditions:

1. $\psi$ is monotone increasing and continuous; 
2. $\psi(t) = 0$ if and only if $t = 0$.
2 Main Results

In this section we will give a fixed point theorem for multivalued mappings which satisfies \((\varphi, \psi)\) – contraction condition.

**Theorem 2.1.** Let \(X_w\) be a complete modular metric space, \(\emptyset \neq M \subseteq X_w\) and \(T : M \rightarrow K(M)\) be a multivalued mapping satisfies following conditions with Fatou Property:

\[
\psi(H_w(Tx,Ty)) \leq \psi(M_w(x,y)) - \varphi(N_w(x,y)) \tag{3.1}
\]

for all \(x, y \in M\) and \(x \neq y\); where

\[
M_w(x, y) = \max \left\{ w_1(x, y), \delta_1(x, Tx), \delta_1(y, Ty), \frac{1}{2}(\delta_2(x, Ty) + \delta_2(y, Tx)) \right\} \\
N_w(x, y) = \min \left\{ w_1(x, y), \delta_1(x, Tx), \delta_1(y, Ty), \frac{1}{2}(\delta_2(x, Ty) + \delta_2(y, Tx)) \right\}
\]

Let \(\varphi(t) : [0, \infty) \rightarrow (0, \infty), \varphi(t) > 0\) is semicontinuous for all \(t > 0\) and discontinuous at \(t = 0\);

\[
\psi(t) : [0, \infty) \rightarrow [0, \infty) \text{ be altering distance function.}
\]

Then \(T\) has a fixed point in \(M \subseteq X_w\); where \(w_1(x_0, x_1) < \infty\) for some \(x_0, x_1 \in X_w\) and \(H_w(A, B)\) is modular Hausdorff metric.

**Proof.** Let \(x_0 \in M\) be arbitrary and \(x_1 \in Tx_0\). Then there is \(x_2 \in Tx_1\) such that

\[
w_1(x_1, x_2) \leq H_w(Tx_0, Tx_1)
\]

Since \(\psi\) is monotone increasing we get

\[
\psi(w_1(x_1, x_2)) \leq \psi(H_w(Tx_0, Tx_1))
\]

If we apply (3.1), we get

\[
\psi(w_1(x_1, x_2)) \leq \psi(H_w(Tx_0, Tx_1)) \leq \psi(M_w(x_0, x_1)) - \varphi(N_w(x_0, x_1)) \tag{3.2}
\]

When we write \(x_3\), instead of \(x_0, x_{2n+1}\) instead of \(x_1\) and \(x_{2n+2}\) instead of \(x_2\) in (2.2) and expanded the inequality we get

\[
\psi(w_1(x_{2n+1}, x_{2n+2})) \leq \psi(H_w(Tx_{2n}, Tx_{2n+1})) \leq \psi(M_w(x_2, x_{2n+1})) - \varphi(N_w(x_{2n}, x_{2n+1})) \tag{3.3}
\]

where

\[
M_w(x_{2n}, x_{2n+1}) = \max \left\{ w_1(x_{2n}, x_{2n+1}), \delta_1(x_{2n}, Tx_{2n}), \delta_1(x_{2n+1}, Tx_{2n+1}) \right\} \\
\delta_1(x_{2n},Tx_{2n}) = \inf_{x_{2n+1} \in Tx_{2n}} w_1(x_{2n}, x_{2n+1}) \\
\delta_1(x_{2n+1},Tx_{2n+1}) = \inf_{x_{2n+2} \in Tx_{2n+1}} w_1(x_{2n+1}, x_{2n+2}) \\
\delta_2(x_{2n},Tx_{2n+1}) = \inf_{x_{2n+2} \in Tx_{2n+1}} \{ w_1(x_{2n+1}, x_{2n+2}) \} \\
\delta_2(x_{2n+1},Tx_{2n}) = \inf_{x_{2n} \in Tx_{2n}} \{ w_1(x_{2n+1}, x_{2n+1}) \} = 0
\]

is satisfied and (2.3) is equal to

\[
M_w(x_{2n}, x_{2n+1}) = \max \left\{ w_1(x_{2n}, x_{2n+1}), w_1(x_{2n+1}, x_{2n+2}), \frac{1}{2}w_2(x_{2n}, x_{2n+2}) \right\}
\]

\[
M_w(x_{2n}, x_{2n+1}) = \max \left\{ w_1(x_{2n}, x_{2n+1}), w_1(x_{2n+1}, x_{2n+2}), \frac{1}{2}(w_1(x_{2n+1}, x_{2n+2}) + w_1(x_{2n+1}, x_{2n+2})) \right\}
\]
Let us assume that 
\[ M_w(x_{2n}, x_{2n+1}) = w_1(x_{2n+1}, x_{2n+2}) \]
If we consider this assumption in equation (3.3) we conclude that 
\[ \psi(w_1(x_{2n+1}, x_{2n+2})) = \psi(w_1(x_{2n+1}, x_{2n+2})) - \varphi(N_w(x_{2n}, x_{2n+1})) \] (3.4)
Since \( \varphi > 0 \)
\[ \psi(w_1(x_{2n+1}, x_{2n+2})) < \psi(w_1(x_{2n+1}, x_{2n+2})) \] (3.5)
Since \( \psi \) is monotone increasing we get 
\[ w_1(x_{2n+1}, x_{2n+2}) < w_1(x_{2n+1}, x_{2n+2}) \] (3.6)
But this is a contradiction. Then we conclude that 
\[ M_w(x_{2n}, x_{2n+1}) = w_1(x_{2n}, x_{2n+1}) \]
Thus \( (w_1(x_{2n}, x_{2n+1})) \) is a monotone decreasing sequence. Since \( K(M) \) is compact, it is closed and bounded and it is bounded from above. That is for \( r > 0 \), we get 
\[ \lim_{n \to \infty} w_1(x_{2n}, x_{2n+1}) = r \]
If we take the limit for \( n \to \infty \) in (3.3), we get 
\[ \lim_{n \to \infty} \psi(w_1(x_{2n+1}, x_{2n+2})) \leq \lim_{n \to \infty} \psi(M_w(x_{2n}, x_{2n+1})) - \lim_{n \to \infty} \varphi(N_w(x_{2n}, x_{2n+1})) \]
for all \( r > 0 \). Since \( \psi \) is continuous 
\[ \psi(\lim_{n \to \infty} w_1(x_{2n+1}, x_{2n+2})) \leq \psi(\lim_{n \to \infty} w_1(x_{2n+1}, x_{2n+1})) - \lim_{n \to \infty} \varphi(N_w(x_{2n}, x_{2n+1})) \]
\[ \psi(r) \leq \psi(r) - \lim_{n \to \infty} \varphi(N_w(x_{2n}, x_{2n+1})) \]
From the definition of \( \varphi \), \( \varphi(N_w(x_{2n}, x_{2n+1})) \neq 0 \). So we get 
\[ \psi(r) < \psi(r) \]
But this is a contradiction. Thus we get \( r = 0 \). That is 
\[ \lim_{n \to \infty} w_1(x_{2n}, x_{2n+1}) = 0 \]
Now let us show that \( (x_n) \) is a Cauchy sequence. To show that is sufficient to show that the subsequence \( (x_{2n}) \) is a Cauchy sequence. Assume that \( (x_{2n}) \) is not Cauchy, then there is 
\[ w_1(x_{2n}, x_{2m}) \geq \varepsilon \]
for \( \exists \varepsilon > 0; m > n > n_0(\varepsilon) \) and \( m \) and \( n \) are the first numbers that satisfies the inequality above.
We can write 
\[ \psi(w_1(x_{2n}, x_{2m})) \leq \psi(H_w(Tx_{2n-1}, Tx_{2m-1})) \leq \psi(M_w(x_{2n-1}, x_{2m-1})) - \varphi(N_w(x_{2n-1}, x_{2m-1})) \]
for \( x_{2n} \in T(x_{2n-1}) \) and \( x_{2m} \in T(x_{2m-1}) \).
\[ M_w(x_{2n-1}, x_{2m-1}) = \max \left\{ \frac{w_1(x_{2n-1}, x_{2m-1})}{2}, \frac{\delta_1(x_{2n-1}, x_{2m-1})}{2}, \frac{\delta_1(x_{2m-1}, x_{2m-1})}{2}, \delta_2(x_{2n-1}, x_{2m-1}) + \delta_2(x_{2m-1}, x_{2n-1}) \right\} \]
\[ = \max \left\{ w_1(x_{2n-1}, x_{2m-1}), w_1(x_{2m-1}, x_{2n}), w_1(x_{2m-1}, x_{2n}), \frac{w_2(x_{2n-1}, x_{2m}) + w_2(x_{2m-1}, x_{2n})}{2} \right\} \]
\[ w_1(x_{2m-1}, x_{2n-1}) \leq w_1(x_{2m-1}, x_{2n}) + w_2(x_{2n}, x_{2n-1}) \]

Since metric modular is monotone decreasing, there is \( 2\varepsilon > 0 \) for \( \frac{1}{2} > 0 \) such that
\[ w_2(x_{2m-1}, x_{2n}) < 2\varepsilon \]

Now if we take the limit for \( m, n \to \infty \) we get
\[ w_1(x_{2m-1}, x_{2n-1}) \leq 2\varepsilon \]

From the main property of metric modular we can write the inequality below;
\[ w_2(x_{2m-1}, x_{2n}) \leq w_1(x_{2m-1}, x_{2n}) < \varepsilon \]

When we write these inequalities we get
\[ M_w(x_{2m-1}, x_{2n-1}) = \max \left\{ 2\varepsilon, 0, 0, \frac{1}{2}(2\varepsilon + \varepsilon) \right\} = 2\varepsilon \]

And if we write these results in (3.1) we get
\[ \psi(2\varepsilon) \leq \psi(2\varepsilon) - \varphi(N_w(x_{2m-1}, x_{2m-1})) \]

Since \( \varphi(N_w(x_{2m-1}, x_{2m-1})) \) can't equal to zero this inequality turns into
\[ \psi(2\varepsilon) < \psi(2\varepsilon) \]

which is a contradiction. Hence our assumption is wrong and \( (x_{2n}) \) is a Cauchy sequence.

Now let us show the existence of the fixed point. Let us assume otherwise, that is \( \bar{x} \) is not a fixed point of \( T \), while \( (x_n) \to \bar{x} \).

Since \( K(M) \) is compact, it is also closed and bounded. So there is a \( \bar{x} \in K(M) \subseteq X_w \) such that \( (x_n) \to \bar{x} \). Then from the Fatou property we get
\[ \delta_1(\bar{x}, T\bar{x}) \leq \liminf_{n \to \infty} w_1(x_{n+1}, T\bar{x}) = \liminf_{n \to \infty} w_1(Tx_n, T\bar{x}) \leq \lim_{n \to \infty} H_w(Tx_n, T\bar{x}) \]

Hence we can write that
\[ \psi(\delta_1(\bar{x}, T\bar{x})) \leq \psi(H_w(Tx_n, T\bar{x})) \leq \psi(M_w(x_n, \bar{x}) - \varphi(N_w(x_n, T\bar{x})) \]

\[ M_w(x_n, \bar{x}) = \max \left\{ w_1(x_n, \bar{x}), \delta_1(x_n, Tx_n), \delta_1(\bar{x}, T\bar{x}), \frac{1}{2}(\delta_2(x_n, \bar{x}) + \delta_2(\bar{x}, T\bar{x})) \right\} \]

\[ \lim_{n \to \infty} M_w(x_n, \bar{x}) = \max \left\{ w_1(\bar{x}, \bar{x}), \delta_1(\bar{x}, T\bar{x}), \delta_1(\bar{x}, T\bar{x}), \frac{1}{2}(\delta_2(\bar{x}, T\bar{x}) + \delta_2(\bar{x}, T\bar{x})) \right\} \]

\[ = \max \left\{ 0, 0, \delta_1(\bar{x}, T\bar{x}), \frac{1}{2}(\delta_2(\bar{x}, T\bar{x})) \right\} \]
From the main property of a modular metric we can write

\[ \delta_2(x, T\overline{x}) \leq \delta_1(x, T\overline{x}) \]

Hence we get

\[ \lim_{n \to \infty} M_w(x_n, \overline{x}) = \delta_1(x, T\overline{x}) \]

From this equality we find that

\[ \psi(\delta_1(x, T\overline{x})) \leq \psi(\delta_1(x, T\overline{x})) - \varphi(N_w(x_n, \overline{x})) \]

From the definition of \( \varphi \) this inequality turns into

\[ \psi(\delta_1(x, T\overline{x})) < \psi(\delta_1(x, T\overline{x})) \]

which is a contradiction. Hence our assumption is wrong and \( \overline{x} \) is a fixed point of \( T\overline{x} \). That is \( \overline{x} \in T\overline{x} \). This completes the proof.

3 Conclusions

In this paper we give \( \varphi - \psi \) contraction for multivalued mappings in modular metric spaces and a fixed point result is shown. We try to expand fixed point theory for modular metric spaces. Authors can develop this results for other spaces like generalised modular metric spaces. Also authors can see [16],[15],[25] and references therein for more information about modular metric spaces.

Competing Interests

Authors have declared that no competing interests exist.

References


© 2020 Turkoglu and Kilinc; This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution and reproduction in any medium, provided the original work is properly cited.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sduarticle4.com/review-history/64013