On the Irrationality and Transcendence of Rational Powers of $e$

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Author’s contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/ARJOM/2021/v17i230277

Editor(s):
(1) Dr. Sakti Pada Barik, Gobardanga Hindu College, India.

Reviewer(s):
(1) Gilbert Makanda, Central University of Technology, South Africa.
(2) K. S. Vidhyaa, Anna Univeristy, India.

Complete Peer review History: http://www.sdiarticle4.com/review-history/67127

Received: 27 February 2021
Accepted: 02 April 2021
Published: 27 April 2021

Abstract

A number that can’t be expressed as the ratio of two integers is called an irrational number. Euler and Lambert were the first mathematicians to prove the irrationality and transcendence of $e$. Since then there have been many other proofs of irrationality and transcendence of $e$ and generalizations of that proof to rational powers of $e$. In this article we review various proofs of irrationality and transcendence of rational powers of $e$ founded by mathematicians over the time.

Keywords: Irrationality; transcendence; Euler’s number.

2020 Mathematics Subject Classification: 11J72, 11J81.

1 Background

The most well known proof of Irrationality of $e$ was proven by Joseph Fourier using proof by contradiction [1]. Before that Euler already wrote the first proof of Irrationality of $e$ using the simple continued fraction expansion of $e$ back in 1737 [2]-[4].

\[ e = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \ddots}}}}} \quad (1.1) \]

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This is an infinite simple continued fraction, which is always irrational. A more simpler proof of this continued fraction was given by Cohn [5]. The proof by contradiction given by Fourier works like this:

Let us assume that $e$ is a rational number and can be expressed as $\frac{p}{q}$, where $p, q$ are integers. Now $e$ can be expressed as:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{q}{n!} + \sum_{n=q+1}^{\infty} \frac{1}{n!}$$

Multiplying both sides by $q!$, we get

$$p(q-1)! = \sum_{n=0}^{q} \frac{q!}{n!} + \sum_{n=q+1}^{\infty} \frac{q!}{n!}$$

Both the LHS and first term of RHS are integer, but the second term is

$$\sum_{n=q+1}^{\infty} \frac{q!}{n!} < \sum_{n=1}^{\infty} \frac{1}{(q+1)n} = \frac{1}{q+1} \left( \frac{1}{1} - \frac{1}{q+1} \right) = \frac{1}{q} < 1 \quad (1.2)$$

which is not an integer. Hence we arrive at a contradiction. MacDivitt [6] gave a proof similar to the above proof, it uses the fact that

$$(b+1)x = 1 + \frac{1}{b+2} + \frac{1}{(b+2)(b+3)} + ... < 1 + \frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + ... = 1 + x \quad (1.3)$$

which proves that $bx < 1$, but that is not possible since both $b$ and $x$ are integers.

Penesi [7], Apostol [8] proved this by proving $e^{-1}$ instead of proving $e$ irrational. Note that the expansion of $e^{-1}$ is

$$e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

Let us define the truncated part of the expansion as $t_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$. Therefore we can write

$$e^{-1} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} + \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!}$$

Let us assume $e^{-1} = \frac{m}{n}$. Multiplying both sides of the previous equation by $n!$, we get the LHS as an integer, and the first term of the RHS as an integer. Therefore we must have

$$n! \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!}$$

as an integer. But this also satisfies

$$0 \leq \left| n! \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!} \right| \leq \frac{n!}{(n+1)!} \leq 1 \quad (1.4)$$

We therefore arrive at a contradiction and hence $e^{-1}$ is irrational. Higher powers of $e$ were subsequently also proven to be irrational. The irrationality of $e^2$ was proven in [9], of $e^3$ in [10], and of $e^4$ in [11].
2 Proof using Niven’s Polynomials

A more generalized result where the power is a rational number was proven by Niven in 1985. It is first proved by Ivan Niven [12] that \( e^{r/q} \) is an irrational number using Niven’s Polynomials of the form \( \frac{n! (1 - x)^n}{n^q} \), which can also be used to prove that \( \pi \) is an irrational number. A similar proof was also given by Aigner [13], Beatty [14] and Eugeni [15].

Let us define \( f : [0, 1] \rightarrow \mathbb{R} \), \( f(x) = \frac{x^p (1 - x)^n}{n!} \) then we have \( f(x) = f(1 - x) \) and \( 0 \leq f(x) < \frac{1}{n!} \). We also note that these functions satisfy

\[
 f^{(i)}(0) , f^{(i)}(1) \in \mathbb{Z}, \quad j \geq 0
\]

Let us assume that \( e^p = \frac{a}{b} \), where \( p \) is an integer. Let us define another function \( F \) as

\[
 F = p^{2n} f - p^{2n-1} f' + p^{2n-2} f'' - \cdots + f^{(2n)}
\]

This function satisfies

\[
 F' + pF = p^{2n+1} f
\]

Multiplying both sides by \( be^{px} \) and then integrating we get

\[
 b \left[ e^{px} F(x) \right]_0^1 = b \int_0^1 p^{2n+1} e^{px} f(x) dx \rightarrow 0^+
\]

as \( n \rightarrow \infty \). Now note that the LHS is \( b[e^{p}F(1) - F(0)] = aF(1) - bF(0) \) which must belong to \( \mathbb{Z}^+ \). But that is not possible, therefore we arrive at a contradiction. Now as \( e^p \) is an integer, any root of that number \( (e^p)^{\frac{1}{2}} \) will also be an irrational number. Another beautiful proof using polynomials of similar form was stated by Joe Mercer [16]. Let us take the two integrals:

\[
 I_n = \frac{1}{n!} \int_0^\infty (x-p)^n e^{-x} dx, \quad J_n = \frac{1}{n!} \int_0^\infty (x+p)^n e^{-x} dx
\]

As the polynomials inside the integral (2.2) \( (x-p)^n, (x+p)^n \) have integer coefficients and the least power of \( x \) is \( n \), we must have both \( I_n \) and \( J_n \) as integer as \( \int_0^\infty x^k e^{-x} dx = k! \). Let us assume that \( e^p = \frac{a}{b} \). Let us multiply \( e^p \) by \( gI_n \). We then have

\[
 ge^p I_n = \frac{ge^p}{n!} \int_0^\infty (x-p)^n e^{-x} dx + gJ_n \int_0^\infty (x+p)^n e^{-(x-p)} dx
\]

\[
 = \frac{ge^p}{n!} \int_0^\infty (x-p)^n e^{-x} dx + g \frac{e^p}{n!} \int_0^\infty (u+u)^n e^{-u} du
\]

\[
 = \frac{ge^p}{n!} \int_0^\infty (x-p)^n e^{-x} dx + gJ_n
\]

Now note that since \( x|x-p| \leq \frac{a^2}{4} \) in \( [0, p] \) and \( 0 < e^{-x} \leq 1 \), we have

\[
 \left| \frac{ge^p}{n!} \int_0^\infty (x-p)^n e^{-x} dx \right| \leq \frac{me^p p^{2n}}{4^n n!}
\]

Now since factorial grow faster than exponential, we can choose an \( n \) such that \( n! > me^p (\frac{a^2}{4})^n \). Also note that \( \int_0^\infty (x-p)^n e^{-x} dx \) will be never be 0 since \( (x-p)^n e^{-x} \) never changes in sign in \( [0, p] \). Thus we have due to (2.3)

\[
 ge^p I_n = \epsilon_n + gJ_n
\]

Here \( 0 < |\epsilon_n| < 1 \) if \( n! > me^p (\frac{a^2}{4})^n \). We therefore arrive at a contradiction since RHS of the above mentioned equation can’t be an integer. We hence proved that there is no integral multiple of \( e^p \) which can be an integer.
3 Proof using Continued Fractions

The proof stated in the section is discussed by Ghosh [17, 18]. We start with the Continued Fraction Expansion of the hyperbolic tanh function discovered by Gauss [19, 20]

\[
\tanh z = \frac{z}{1 + \frac{z^2}{3 + \frac{z^2}{5 + \frac{z^2}{7 + \cdots}}}}.
\]

We also know that the hyperbolic tanh function is related to the exponential function with the following formula

\[
\tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}}.
\]

Putting \(\frac{z}{x}\) in place of \(z\) in the previous equation we get

\[
\frac{e^{\frac{z}{x}} - e^{-\frac{z}{x}}}{e^{\frac{z}{x}} + e^{-\frac{z}{x}}} = \frac{\left(\frac{z}{x}\right)}{1 + \frac{\left(\frac{z}{x}\right)^2}{3 + \frac{\left(\frac{z}{x}\right)^2}{5 + \cdots}}},
\]

This continued fraction can be simplified into

\[
\frac{e^{\frac{z}{x}} - e^{-\frac{z}{x}}}{e^{\frac{z}{x}} + e^{-\frac{z}{x}}} = \frac{x}{y + \frac{x^2}{3y + \frac{x^2}{5y + \cdots}}},
\]

This equation can be further be simplified as

\[
1 + \frac{2}{e^{\frac{z}{x}} - 1} = y + \frac{x^2}{3y + \frac{x^2}{5y + \cdots}}.
\]

Some algebraic manipulation, yields a continued fraction expansion of \(e^{x/y}\)

\[
e^{x/y} = 1 + \frac{2x}{2y - x + \frac{x^2}{6y^2 + \frac{x^2}{10y^2 + \frac{x^2}{14y^2 + \cdots}}}},
\]

which is an infinite continued fraction. Legendre found necessary and sufficient conditions for the convergence of the continued fraction in following theorem. The conditions were first published by Chrystal [21].

**Theorem 3.1.** The necessary and sufficient condition that the continued fraction

\[
\frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \cdots}}} = \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \cdots}}}
\]

is irrational is that the values \(a_i, b_i\) are all positive integers, and there is an integer \(n\) such that \(|a_i| > |b_i|\) for all \(i\) greater than \(n\).
In the continued fraction of $e^{x/y} - 1$, we have derived $a_i, b_i$ are equals to $2(2^i - 1)$, $x^2$ except when $i = 1$. Therefore we have $|a_i| > |b_i|$ for all $i > \frac{x^2}{2^i - 1}$. Hence we have proved that $e^{x/y} - 1$ is irrational which in turn means $e^{x/y}$ is irrational, where $x, y$ are integers.

4 Transcendence of Rational Powers of $e$

A transcendental number is a number that cannot be expressed as the root of a non-zero polynomial with all its coefficients being rational. Note that an irrational number does not necessarily have to be a transcendental number. Square-root of any non-square integer is an irrational number but not a transcendental number. A number that is not transcendental is called an algebraic number.

It was first Hermite [22, 23] who proved that $e$ is transcendental. This results were further extended by Lindemann who proved that $e^\alpha$ is transcendental, given $\alpha$ is a non-zero transcendental number [24, 25]. Using this he also proved that $\pi$ is transcendental, since $e^{i\pi} = -1$, which is a real number. Weierstrass generalized this proof [26] to give the well known Lindemann-Weierstrass theorem. Hilbert [27], Gordan [28] simplified this proof. A similar theorem establishing that $a^b$ is a transcendental number given that $a$ is an algebraic number satisfying $a \neq 0, 1$ and $b$ is an algebraic number which is irrational but not transcendental was proved by Gelfond [29] known as Gelfond-Schneider theorem. All of these theorems are generalized further by Schanuel’s conjecture [31]. Bernard [32] proved the transcendence of $e$ using multivariate and symmetric Polynomials.

In this article we shall only discuss about the transcendence of of rational powers of $e$. To prove that $e^v$ is transcendental, where $v$ is a rational number, let us assume that $e^v$ is algebraic and satisfies

$$c_0 + c_1 e^v + c_2 e^{2v} + ... + c_n e^{nv} = 0 \quad (4.1)$$

where all coefficients $c_t(0 \leq t \leq n)$ are integers with $c_0, c_n$ being non-zero. We now employ a function which is an extension of Niven’s Polynomials:

$$f_k(x) = v^{2k+2} x^k[(x-1)...(x-n)]^{k+1}$$

Note that the least power of $x$ in $f_k(x)$ is $k$, but the least power of $x$ in $f_k(x+a)$, where $a$ is $0 < a \leq n$ is $k+1$. Multiplying both sides of (4.1) by $\int_0^\infty f_k e^{-vx} dx$, we get the following equation:

$$\sum_{t=0}^n c_t e^{tv}(\int_0^\infty f_k e^{-vx} dx) = 0$$

The LHS can be divided into two parts $P, Q$ such that $P + Q = 0$

$$P = \sum_{t=0}^n c_t e^{tv}(\int_0^\infty f_k e^{-vx} dx) \quad (4.2)$$

$$Q = \sum_{t=1}^n c_t e^{tv}(\int_0^\infty f_k e^{-vx} dx) \quad (4.3)$$

We now derive two lemmas to prove the transcendence of $e$.

Lemma 4.1. $\frac{P}{Q}$ is a positive integer

Note that every term in $P$ will contain sum of integer multiples of integrals of the form

$$\int_0^\infty x^t e^{-vx} dx = \frac{j!}{v^{j+1}} \quad (4.4)$$
which is the value of gamma function at integer points. Note that the integrand $c_t e^{tv} \int_0^\infty f_k e^{-vx} dx$
for every $t$ satisfying $0 < t \leq n$ is a sum of terms whose lowest and highest power of $x$ is $k+1, 2k+1$
respectively, multiplied with $e^{-vx}$ integrated from $0$ to $\infty$ after substituting $x$ for $x+a$ since

$$c_t e^{tv} \int_0^\infty f_k e^{-vx} dx = c_t e^{tv} \int_0^\infty f_k(x + t)e^{-v(x+t)} dx = c_t \int_0^\infty f_k(x + t)e^{-vx} dx$$

Therefore $P$ can be written as

$$P = c_0 e^0 \left( \int_0^\infty f_k e^{-vx} dx \right) + \sum_{t=1}^n c_t \sum_{j=k+1}^{2k+1} A_{j-k,t} v^{2k-j+1} (v^{j+1} \int_0^\infty x^j e^{-vx} dx) \quad (4.5)$$

Substituting (4.4) in (4.5), we get

$$P = c_0 e^0 \left( \int_0^\infty f_k e^{-vx} dx \right) + \sum_{t=1}^n c_t \sum_{j=k+1}^{2k+1} A_{j-k,t} v^{2k-j+1} j!$$

Here $A_{j-k,t}$ refers to the integer coefficient of $x^j$ in $\frac{1}{(x+t)^{j-k}}$. All the terms in the second part of
RHS of (4.5) are divisible by $(k+1)!$. Therefore after division by $k!$, it must be also divisible by
$(k+1)$! The first part of RHS of (4.5) can be expressed as

$$c_0 e^0 \left( \int_0^\infty f_k e^{-vx} dx \right) = \int_0^\infty v^{2k+2} \left( \left\langle -1^n n! \right\rangle^{k+1} e^{-vx} x^k + \ldots \right) dx$$

The higher order terms in RHS shall be divisible by $(k+1)$. Therefore we get

$$\frac{1}{k!} c_0 \left( \int_0^\infty f_k e^{-vx} dx \right) \equiv c_0 \left( \left\langle -1^n n! \right\rangle^{k+1} v^{k+1} \right) \neq 0 \mathrm{mod} (k+1) \quad (4.6)$$

We see that $\frac{P}{k!}$ is not divisible by $k+1$, if it is a prime greater than $n, |c_0|$. But since $P$ is divisible
by $k!$, $\frac{P}{k!}$ cannot be zero.

**Lemma 4.2.** There exists some $k$ such that $|\frac{Q}{k!}| < 1$

Let us start with two continuous functions $g(x), f(x)$

$$g(x) = v^2 x(x-1) \ldots (x-n) \quad (4.7)$$

$$f(x) = v^2 (x-1) \ldots (x-n) e^{-vx} \quad (4.8)$$

Since both of them are continuous functions, they are bounded in the interval $[0, n]$. Let the upper bounds be $b_1, b_2 > 0$ respectively. Therefore $f_k e^{-vx} = g(x)^k f(x)$ is also bounded by $b_1^k b_2$ in the interval $[0, n]$. Each of the integrals are themselves bounded since

$$\left| \int_0^n f_k e^{-vx} dx \right| \leq \int_0^n \left| f_k e^{-vx} \right| dx \leq \int_0^n b_1^k b_2 dx \equiv (n-t)b_1^k b_2$$

Therefore the sum $Q$ is itself bounded as

$$|Q| < nb_2^k b_2 (c_0 + c_1 e^v + c_2 e^{2v} + \ldots + c_n e^{nv}) = b_1^k w \quad (4.9)$$

Here $w = nb_2 (c_0 + c_1 e^v + c_2 e^{2v} + \ldots + c_n e^{nv})$ is independent of $k$. Therefore we get

$$\frac{|Q|}{k!} < \frac{w b_1^k}{k!} \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty$$

**Theorem 4.3.** $e^v$ is a transcendental number.
We note that
\[
\frac{1}{k!} \sum_{t=0}^{n} c_t e^{tv} \left( \int_{0}^{\infty} f_k e^{-v^x} dx \right) = \frac{1}{k!} (P + Q) = 0
\]
But \( P_k \) is a positive integer whereas \( Q_k \) is a very small real number close to zero. The sum of \( P_k \) and \( Q_k \) can never be zero. Therefore our original assumption is wrong. Hence \( e^v \) does not satisfies
\[
c_0 + c_1 e^v + c_2 e^{2v} + \ldots + c_n e^{nv} = 0
\]
where all coefficients \( c_t (0 \leq t \leq n) \) are integers with \( c_0, c_n \) being non-zero. Therefore \( e^v \) is transcendental number. Since any \( n^{th} \) root of \( e^v \) is also transcendental number, we must have \( e^{p/q} \) a transcendental number for any rational number \( p/q \).

5 Conclusion
In this article we reviewed various proofs of irrationality and transcendence of rational powers of \( e \) founded by mathematicians over the time.

Competing Interests
Author has declared that no competing interests exist.

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