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Edge-Complement Graphs – Another Approach

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Authors' contributions

This work was a combined work by both the authors. Authors SK and HEV defined the characteristic function on a graph and proved some results on spanning sub graphs using the newly defined characteristic function. Both the authors read and approved the final manuscript.

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Abstract

The collection of edge complement spanning subgraphs of a simple graph is an abelian group with respect to the symmetric difference operation.

Keywords: *Edge-Complement graph; spanning graph; commutative algebra; symmetric difference; characteristic function.*

1 Introduction

The authors [1] introduced the concept of Edge complement graph and authors [2] proved some results related to these concepts. In [3], [4] and [5] similar concepts are explained with respect to graphs. In [6], it is proved that collection of subsets of a set is an abelian group, with respect to the symmetric difference(Δ). In this paper, the authors extend this idea on to the Edge Complement Spanning subgraph of a simple graph while defining the characteristic function on graphs and proved related results on characteristic functions on graphs [7-9]

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1.1 Definition

A graph G with vertices *p* and edges *q* is called a *(p, q)* graph, where *p* and *q* are respectively known as the order and the size of the graph. The *(p, q)* graph with $q = 0$, is called an empty graph and is denoted by Φ. H is a spanning subgraph of a graph G if $V(H) = V(G)$

1.2 Definition

Let $G(V_G, E_G)$ be a graph and $H(V_H, E_H)$ be a spanning subgraph of G. The edge complement of H in G is obtained by deleting all the edges of H from G and is denoted by $H^c_{E_G}$. For simplicity, we refer H^c for $H^c_{E_G}$.
From the definition of edge complement graph, we have the following results:

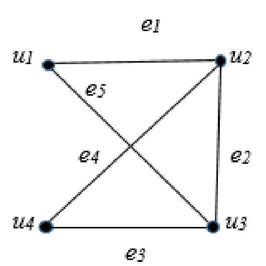
$$V(H) = V(H^c) = V(G)$$
$$E(H) \subseteq E(G),\ E(H^c) \subseteq E(G)$$
$$E(H \cup H^c) = E(G) \text{ and } E(H \cap H^c) = \phi$$

So, the edge complement of subgraphs of a graph satisfies the structure property.

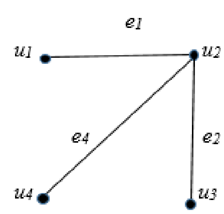
1.3 Definition

The symmetric difference of the spanning graphs $H(V_H, E_H)$ and $K(V_K, E_K)$ of $G(V_G, E_G)$ is a spanning graph $H\Delta K$ of G whose edge set is $E_{H\Delta K} = E_H \cup E_K - E_H \cap E_K$

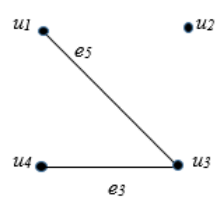
1.4 Example



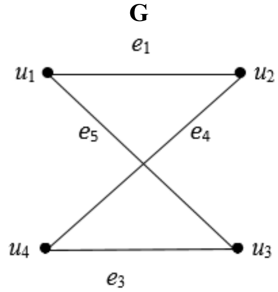
G



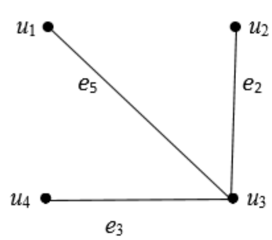
H



H^c



K



HΔK

1.5 Definition

Let G be any simple graph, H be a edge complement spanning subgraph of G. The characteristic function χ on H in G is defined for each $u_i u_j \in E(G)$ by

$$\chi_H(u_i u_j) = \begin{cases} 1 \ \ if \ \ u_i u_j \in E(H) \\ 0 \ \ if \ \ u_i u_j \notin E(H) \end{cases}$$

2 EDGE Complement Graph

2.1 Lemma

Let H, K be the spanning subgraphs of G then $\chi_H = \chi_K$ if and only if $H = K$.

Proof:

For each $u_i u_j \in E(H)$,

$$\chi_H(u_i u_j) = \begin{cases} 1 \ if \ u_i u_j \in E(H) \\ 0 \ if \ u_i u_j \notin E(H) \end{cases}$$
$$= \begin{cases} 1 \ if \ u_i u_j \in E(K) \\ 0 \ if \ u_i u_j \notin E(K) \end{cases}$$
$$= \chi_K(u_i u_j)$$

Therefore, for each $u_i u_j \in E(H)$ and $\chi_H = \chi_K$, then $\chi_K(u_i u_j) = 1$. So $E(H) \subseteq E(K)$. Similarly, $E(K) \subseteq E(H)$. Hence, $H = K$.

2.2 Lemma

$$\chi_{H^c} = 1 - \chi_H$$

Proof:

For each $u_i u_j \in E(G)$,

$$\chi_{H^c}(u_i u_j) = \begin{cases} 1 \ if \ u_i u_j \in E(H^c) \\ 0 \ if \ u_i u_j \notin E(H^c) \end{cases}$$
$$= \begin{cases} 1 - 0 \ if \ u_i u_j \in E(H^c) \\ 1 - 1 \ if \ u_i u_j \notin E(H^c) \end{cases}$$
$$= 1 - \begin{cases} 0 \ if \ u_i u_j \notin E(H) \\ 1 \ if \ u_i u_j \in E(H) \end{cases}$$
$$= 1 - \begin{cases} 1 \ if \ u_i u_j \in E(H) \\ 0 \ if \ u_i u_j \notin E(H) \end{cases}$$
$$= (1 - \chi_H)(u_i u_j)$$

Hence, $\chi_{H^c} = 1 - \chi_H$

2.3 Lemma

$$\chi_{H \cap K} = \chi_H \cdot \chi_K$$

Proof:

For each $u_i u_j \in E(G)$,

$$\chi_{H \cap K}(u_i u_j) = \begin{cases} 1 \ if \ u_i u_j \in E(H \cap K) \\ 0 \ if \ u_i u_j \notin E(H \cap K) \end{cases}$$
$$= \begin{cases} 1 \ if \ u_i u_j \in E(H) \ and \ u_i u_j \in E(K) \\ 0 \ if \ u_i u_j \notin E(H) \ or \ u_i u_j \notin E(K) \end{cases}$$
$$= \begin{cases} 1 \ if \ u_i u_j \in E(H) \\ 0 \ if \ u_i u_j \notin E(H) \end{cases} \cdot \begin{cases} 1 \ if \ u_i u_j \in E(K) \\ 0 \ if \ u_i u_j \notin E(K) \end{cases}$$
$$= \chi_H(u_i u_j) \cdot \chi_K(u_i u_j)$$
$$= (\chi_H \cdot \chi_K)(u_i u_j)$$

Hence, $\chi_{H \cap K} = \chi_H \cdot \chi_K$

Obviously when $K = \Phi$, $\chi_{H\cap\Phi} = 0$, shows that $\emptyset$ is the identity and $\chi_{H\cap H^c} = \chi_\emptyset$ follows that H^c is the inverse of H with respect to ∩. Also $\chi_{(H\cap K)\cap M} = \chi_{H\cap K} \cdot \chi_M = \chi_H \cdot \chi_K \cdot \chi_M = \chi_H \cdot \chi_{K\cap M} = \chi_{H\cap(K\cap M)}$ apparently shoes the associative property and evidently $\chi_{H\cap K} = \chi_H \cdot \chi_K = \chi_K \cdot \chi_H = \chi_{K\cap H}$ shows the commutative property.

2.4 Lemma

$$\chi_{(H\cup K)^c} = \chi_{H^c \cap K^c}$$

Proof

For each $u_iu_j \in E(G)$,

$$\chi_{(H\cup K)^c}(u_iu_j) = \begin{cases} 1 \ if \ u_iu_j \in E(H\cup K)^c \\ 0 \ if \ u_iu_j \notin E(H\cup K)^c \end{cases}$$

$$= \begin{cases} 1 \ if \ u_iu_j \notin E(H\cup K) \\ 0 \ if \ u_iu_j \in E(H\cup K) \end{cases}$$

$$= \begin{cases} 1 \ if \ u_iu_j \notin E(H) \ and \ u_iu_j \notin E(K) \\ 0 \ if \ u_iu_j \in E(H) \ or \ u_iu_j \in E(K) \end{cases}$$

$$= \begin{cases} 1 \ if \ u_iu_j \in E(H^c) \ and \ u_iu_j \in E(K^c) \\ 0 \ if \ u_iu_j \notin E(H^c) \ or \ u_iu_j \notin E(K^c) \end{cases}$$

$$= \begin{cases} 1 \ if \ u_iu_j \in E(H^c \cap K^c) \\ 0 \ if \ u_iu_j \notin E(H^c \cap K^c) \end{cases}$$

$$= \chi_{H^c \cap K^c}(u_iu_j)$$

$$\therefore \ \chi_{(H\cup K)^c} = \chi_{H^c \cap K^c}$$

Hence

$$\chi_{(H_1\cup H_2 \cup \ldots \cup H_n)^c} = \chi_{H_1{}^c \cap H_2{}^c \cap \ldots \cap H_n{}^c}$$

2.5 Lemma

$$\chi_{H\cup K} = \chi_H + \chi_K - \chi_{H\cap K}$$

Proof:

By Lemma 2.2

$\chi_{H\cup K} = 1 - \chi_{(H\cup K)^c}$

$= 1 - \chi_{H^c \cap K^c}$

$= 1 - \chi_{H^c} \cdot \chi_{K^c}$ by Lemma 2.3

$= 1 - (1 - \chi_H) \cdot (1 - \chi_K)$

$= 1 - [1 - \chi_H - \chi_K + \chi_H \cdot \chi_K]$

$= \chi_H + \chi_K - \chi_{H\cap K}$ by Lemma 2.3

2.6 Lemma

$$\chi_{H-K} = \chi_H - \chi_{H\cap K}$$

Proof:

$\chi_{H-K} = \chi_{H\cap K^c}$

$= \chi_H \cdot \chi_{K^c}$

$= \chi_H \cdot \chi_{K^c}$

$= \chi_H \ (1 - \chi_K)$

$= \chi_H - \chi_H \cdot \chi_K$

$= \chi_H - \chi_{H\cap K}$

2.7 Lemma

$$\chi_{H\Delta K} = \chi_H + \chi_K - 2\,\chi_{H\cap K}$$

Proof:

$$\chi_{H\Delta K} = \chi_{(H\cup K)-(H\cap K)}$$
$$= \chi_{H\cup K} - \chi_{(H\cup K)\cap(H\cap K)} \quad \text{by Lemma 2.6}$$
$$= \chi_{H\cup K} - \chi_{(H\cap K)}$$
$$= \chi_H + \chi_K - \chi_{H\cap K} - \chi_{(H\cap K)}$$
$$= \chi_H + \chi_K - 2\,\chi_{H\cap K}$$

Obviously, 1. When $K = \Phi$,

$$\chi_{H\Delta\Phi} = \chi_H + \chi_\Phi - 2\,\chi_{H\cap\Phi} = \chi_H \qquad since\ \chi_\Phi = 0\ \ and\ 2\,\chi_{H\cap\Phi} = 2\,\chi_\Phi = 0$$

χ_Φ is the identity.

2. Consider $\chi_{H\Delta K} = \chi_H + \chi_K - 2\,\chi_{H\cap K}$
$= \chi_H + \chi_K - 2\,\chi_{K\cap H} = \chi_{K\Delta H}$

3. When $H = K$ $\chi_{H\Delta K} = \chi_H + \chi_H - 2\chi_{H\cap H} = 0$

So, H is the inverse of H it’s self with respect to symmetric difference.

4. Consider

$$\chi_{(H_1\,\Delta\,H_2)\,\Delta\,H_3} = \chi_{H_1\Delta H_2} + \chi_{H_3} + (-2)\,\chi_{H_1\Delta H_2\cap H_3}$$
$$= \chi_{H_1} + \chi_{H_2} + (-2)\chi_{H_1\cap H_2} + \chi_{H_3} + (-2)\,\chi_{(H_1\Delta H_2)}\cdot\chi_{H_3}$$
$$= \chi_{H_1} + \chi_{H_2} + \chi_{H_3} + (-2)\chi_{H_1\cap H_2} + (-2)\chi_{H_1}\cdot\chi_{H_3} + (-2)\chi_{H_2}\cdot\chi_{H_3} + (-2)^2\chi_{H_1\cap H_2}\cdot\chi_{H_3}$$
$$= \chi_{H_1} + \chi_{H_2} + \chi_{H_3} + (-2)\chi_{H_1\cap H_2} + (-2)\chi_{H_1\cap H_3} + (-2)\chi_{H_2\cap H_3} + (-2)^2\chi_{H_1\cap H_2\cap H_3}$$
$$= \chi_{H_1} + \chi_{H_2} + \chi_{H_3} + (-2)\chi_{H_2\cap H_3} + (-2)\chi_{H_1}\cdot\chi_{H_2} + (-2)\chi_{H_1}\cdot + (-2)^2\chi_{H_1}\cdot\chi_{H_2\cap H_3}$$
$$= \chi_{H_1} + \chi_{H_2} + \chi_{H_3} + (-2)\chi_{H_2\cap H_3} + (-2)[\chi_{H_1}(\chi_{H_2} + \chi_3 + (-2)\chi_{H_2\cap H_3}]$$
$$= \chi_{H_1} + \chi_{H_2} + \chi_{H_3} + (-2)\chi_{H_2\cap H_3} + (-2)[\chi_{H_1}\cdot\chi_{H_2\Delta H_3}]$$
$$= \chi_{H_1} + \chi_{H_2\Delta H_3} + (-2)\chi_{H_2\cap H_2\Delta H_3}$$
$$= \chi_{H_1\,\Delta\,(H_2\,\Delta\,H_3)}$$

3 Abelian Group on Edge Complement Graph

3.1 Theorem

The collection □(G) of edge-complement spanning subgraphs of a simple graph is a abelian group with respect to graph union ∪ graph intersection ∩ and graph symmetric difference Δ .

4 Conclusion

The authors proved some algebraic results on spanning graphs by defining the characteristic function, which can be extended to the general graphs. These ideas can also be extended to the recently developed concepts of super hyper graphs and n-super hyper graphs.

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Competing Interests

Authors have declared that no competing interests exist.

References

[1] Masami Yasuda, Abulimiti Yiming. A note on properties for a complementary graph and its tree graph. Joual of Discrete Mathematical Sciemces and Cryptography; 2005.

[2] Gupta RP. Proof Techniques in Graph Theory, Acamedic Press, New York; 1969.

[3] Bondy JA, Murty USR. Graph theory with applications. Elsevier, North Holland; 1976.

[4] Inter K. Rana. An introduction to measure and integration. Nasora International Publishing, 2nd edition; 2005.

[5] Fedor V. Fomin, Petr A. Golovach, Dimitrios M. Thilikos, Sub graph Complementation, Algorithmica. 2020;82:1859-1880.

[6] Florentin Smarandache. Neutrosophic sets and systems - University of New Mexico, Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary. 2020;33:290-296.

[7] Suresh Singh G. Krishnapillai karunakaran: Edge measurable graphs. Graph Theory Notes of New York. 2005;XLIX:17-21.

[8] Vijayan A, Sanal S. On Cartesian product of vertex measurable graphs. Journal of Discrete Mathematical Sciences & Crytography. 2012;14(2&3):193 – 210.

[9] Harary F. Graph Theory, Addison-Wesley, Reading Mass; 1969.

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