The Influence of Circulatory Forces on the Stability of Undamped Gyroscopic Systems

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Author’s contribution
The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract
In this paper, the effects of circulatory forces on undamped gyroscopic systems have been studied. The stability or otherwise of arising MGKN systems have been analysed using eigenvalue method and Lyapunov method. Bottema-Lakhdanov-Karapetyan theorem is also employed to further analyze the systems. Examples are given to illustrate the use of these methods in determining the stability or otherwise of undamped gyroscopic circulatory systems.

Keywords: Asymptotic stability; circulatory forces; stability; undamped gyroscopic system.

1 Introduction

Gyroscopic systems are generally stable systems due to the presence of the gyroscopic forces. The introduction of circulatory forces in gyroscopic systems may ensure stability or destabilize an already stable system. The undamped gyroscopic circulatory systems have engaged the attention of scientists for over sixty years now. These systems which are also known as MGKN systems are generally unstable systems. These systems find wide applications in sciences and engineering. In spinning satellites, a body-fixed thruster changes from MGK...
systems to MGKN systems. Practically, this means that the constant matrix of the dynamic stiffness matrix is a
general matrix which can always be split into symmetric and anti-symmetric parts. The MGKN systems are not
conservative systems and can have a type of instability known as flutter instability that does not exist in
conservative systems. MGKN systems also find applications in many other areas of real-life situations such as
economics, sociology, biology, vibration of rotor in hydrodynamic bearing and the squealing disk brake, control
of two-legged walking robots, wear in paper calendars, self-oscillations in aircraft wheels and flutter in
aerospace systems [1-9].

Consider the MGKN system

\[ M \ddot{x} + G \dot{x} + (K + N)x = 0 \]  

System (1) can be transformed using the transformation \( q(t) = M^{-1/2}x(t) \) to have

\[ \ddot{q} + G^{\ast} \dot{q} + (K^{\ast} + N^{\ast})q = 0 \]  

where \( G^{\ast} = M^{-1/2}GM^{-1/2} \) and \( N^{\ast} = M^{-1/2}NM^{-1/2} \) are skew-symmetric matrices and \( K^{\ast} = M^{-1/2}KM^{-1/2} \) is symmetric. The eigenvalues of the matrices \( G^{\ast} \) and \( N^{\ast} \) are either zero or conjugate pairs of
pure imaginary numbers. Systems (1) and (2) are equivalent. The skew-symmetric matrix \( N^{\ast} \) are circulatory
forces or nonconservative positional forces and \( G^{\ast} \) another skew-symmetric matrix are gyroscopic forces. The
symmetric matrix \( K^{\ast} \) are the potential forces. In this work, the stability properties of these systems will be
discussed extensively using various methods. Examples are given to demonstrate the efficacy of the methods.
Section 2 deals with the methodology and stability analysis while section 3 considers the applications.
Conclusion is given in section 4.

2 Methodology and Stability Analysis

In this section, some methods of analyzing the stability or otherwise of undamped gyroscopic circulatory
systems are considered.

2.1 Analysis using eigenvalue approach

Given the linear system (1), the characteristic equation for system (1) assumes the form

\[ \det(\lambda^2 M + \lambda G + (K + N)) = 0 \]

The degree of the characteristic polynomial of system (2) in \( \lambda \) is \( 2n \) and the polynomial can be written in \( \lambda \) as

\[ a_{2n} \lambda^{2n} + a_{2n-1} \lambda^{2n-1} + \cdots + a_2 \lambda^2 + a_1 \lambda + a_0 = 0 \]

The first term \( a_{2n} \lambda^{2n} \) is equal to the product of the principal diagonal elements of the matrix \( \left( \lambda^2 M + \lambda G + (K + N) \right) \). The term \( a_{2n} \) is therefore different from zero for a positive mass matrix.

Recall that the principal diagonal of \( G \) is formed exclusively by zeros \( (G^T = -G) \), the coefficient \( a_{2n-1} \) vanishes, so that the first eventually non-vanishing coefficient is \( a_{2n-2} \). Therefore, according to Hurwitz
criterion, NOT all the eigenvalues of a MGKN system have negative real parts (i.e there are purely imaginary
eigenvalues and /or eigenvalues with positive real parts). Furthermore, it can be stated that at least one
eigenvalue with positive real part will occur. Consequently, MGKN systems have at least one eigenvalue with
positive real part and the trivial solution is unstable.
2.2 Analysis using lyapunov method

Given system (1), the undamped gyroscopic circulatory system in the equivalent form of a first order system below

\[ \dot{q} = Aq \]  

where \( A = \begin{bmatrix} 0 & I \\ -M^{-1}(K + N) & -M^{-1}G \end{bmatrix} \), \( q = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \), 1 is the identity matrix.

The system (3) is stable (asymptotically stable) if there exists a Lyapunov function \( V > 0 \) and the time derivative \( \dot{V} \leq 0 \) (\( \dot{V} < 0 \)). This Lyapunov function is in the form

\[ V = \dot{q}(t)^T P q(t) \]

and the time derivative in the form

\[ \dot{V} = \dot{q}(t)^T (A^*P + PA) q(t) \]

expressed by the matrix Q satisfying the Lyapunov matrix equation

\[ A^*P + PA < 0 \]

or

\[ A^*P + PA = -Q \]  

where P and Q are Hermitian matrices. \( P = P^* > 0 \) and \( Q = Q^* \geq 0 \)

If \( Q = 0 \) equation (4) becomes

\[ A^*P + PA = 0 \]  

The system (2) is marginally stable if and only if equation (5) has a symmetric positive definite solution \( P = P^* \geq 0 \) [10]. P is in the form

\[ P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} \]

The stability of (2) is ensured \( P > 0 \) if and only if the matrices \( P_{11} > 0 \) and

\[ P_{22} - P_{12}^* P_{11}^{-1} P_{12} > 0 \] (i.e both are positive definite) or if and only if \( P_{22} \) and

\[ P_{11} - P_{22} P_{12}^* P_{12} \] are positive definite [11].

2.3 Further analysis

The stability or otherwise of system (2) can further be analysed by considering the result of the relationship between the matrices of gyroscopic forces and the matrices of circulatory forces. According to a well known result, the system (2) is not asymptotically stable, but it is unstable if the \( tr(G^*N^*) \neq 0 \) [12,13,14]. This is true when the degrees of freedom are 2 and 3. Observably, when the degree of freedom is 2, the simultaneous action of circulatory and gyroscopic forces always destabilizes a potential system. If the degree of freedom is greater than 2, the system (2) is generically unstable; it can be Lyapunov stable in the set of measure zero if \( tr(G^*N^*) = 0 \). Additionally, if \( G^* \) and \( N^* \) commute, then, the system is unstable

when \( tr(G^*N^*) = 0 \).
System (2) can also be seen as the outcome of a small perturbation of a gyroscopic system by circulatory forces. The Bottema-Lakhadanov-Karapetyan theorem demonstrates that, similar to the destabilizing effect of full dissipation on gyroscopic stabilization, the nonconservative positional forces generically destroy the marginal stability of gyroscopic systems.

3 Applications

Example 1:

Consider the MGKN system of a two degree of freedom

\[
\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \ddot{q}_1 + \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix} \ddot{q}_2 + \left( \begin{bmatrix} b_1 & b_3 \\ b_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & c \\ -c & 0 \end{bmatrix} \right) q_1 = 0
\]

The characteristic equation of the system is given by

\[
a_1 a_2 \lambda^4 + (g^2 + a_1 b_2 + a_2 b_1) \lambda^2 + 2gc \lambda + (c^2 + b_1 b_2 - b_3^2) = 0
\] (4)

Equation (4) has no term in \( \lambda^3 \), thus according to Hurwitz criterion, not all eigenvalues can have negative real parts. This means that there exists at least one positive eigenvalue, therefore, the system is unstable. Eigenvalues with zero real parts are possible only for \( cg = 0 \), namely only if exclusively even powers of \( \lambda \) are present. This however means that the gyroscopic matrix \( G \) and/or the circulatory matrix \( N \) have to vanish so that we would no longer have a MGKN system.

Example 2:

Consider the system described by eqn (2) with the following matrices

\[
K^* = \begin{bmatrix} l_1 I & 0 \\ 0 & l_2 I \end{bmatrix}, \quad G^* = \begin{bmatrix} 0 & -p & 0 & 0 \\ p & 0 & 0 & 0 \\ 0 & 0 & 0 & -p \\ 0 & 0 & p & 0 \end{bmatrix}, \quad N^* = \begin{bmatrix} 0 & t & 0 & 0 \\ -t & 0 & 0 & 0 \\ 0 & 0 & 0 & -t \\ 0 & 0 & t & 0 \end{bmatrix}
\]

where \( I \) is the 2x2 identity matrix and \( l_1, l_2, p \) and \( t \) are arbitrary real constants. The \( tr (G^* N^*) = 0 \). Therefore, the system (2) is surely unstable since the matrices are pairwisely commutative.

4 Conclusion

Gyroscopic systems are generally stable systems due to the presence of gyroscopic forces. When the constant matrix of dynamic stiffness is split into symmetric and anti-symmetric parts, MGKN systems result. Undamped gyroscopic circulatory systems are generally unstable systems because of the introduction of circulatory forces. Circulatory forces may stabilize or destabilize an already unstable or stable system. The stability or otherwise have been studied using various methods and appropriate conclusions have been drawn. Examples are given to demonstrate the use of these methods in determining the stability or otherwise of these systems.

Competing Interests

Author has declared that no competing interests exist.

References