Existence, Nonexistence and Multiplicity Results for Positive Radial Solutions of $n$–dimensional Elliptic System

Yalin Shen$^{1*}$

$^1$School of Applied Science, Beijing Information Science and Technology University, Beijing, 100192, PR China.

Author’s contribution
The sole author designed, analysed, interpreted and prepared the manuscript.

Article Information
DOI: 10.9734/ARJOM/2021/v17i630307

Editor(s):
(1) Dr. Sheng Zhang, Bohai University, China.

Reviewers:
(1) M. Vijayalakshmi, India.
(2) Nivetha Martin, India.

Complete Peer review History: http://www.sdiarticle4.com/review-history/72817

Received: 20 June 2021
Accepted: 26 August 2021

Abstract

Aims/ Objectives: In this paper, we study the existence, nonexistence and multiplicity of positive solutions to the $n$–dimensional elliptic system

\[
\begin{align*}
\Delta u(x) + \Lambda a(|x|) f(u(x)) &= 0 \quad \text{in } \Omega, \\
u(x) &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where $\Omega = \{x \in \mathbb{R}^n : R_1 < |x| < R_2\}, R_1 > R_2 > 0, n \geq 2$, $a(|x|)$ is allowed to change sign on $[R_1, R_2]$, \(x = (x_1, x_2, \ldots, x_n)^T\), \(u(x) = (u_1(x), \ldots, u_k(x), \ldots, u_n(x))^T\), \(\Delta u(x) = (\Delta u_1(x), \ldots, \Delta u_k(x), \ldots, \Delta u_n(x))^T\), \(\Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_k, \ldots, \lambda_n\}\), \(a(|x|) = \text{diag}\{a_1(x), \ldots, a_k(x), \ldots, a_n(x)\}\), \(f(x) = (f_1(x), \ldots, f_k(x), \ldots, f_n(x))^T\). In previous articles, elliptic systems have been widely studied, but there is relatively little research on $n$–dimensional elliptic systems. We are very interested in this subject and want to study it. We give new conclusions on the existence, nonexistence and multiplicity of positive solutions for the $n$–dimensional elliptic system.

*Corresponding author: E-mail: 18810876062@163.com;
Study Design: Study on the existence, nonexistence and multiplicity of positive solutions.

Place and Duration of Study: School of Applied Science, Beijing Information Science & Technology University, September 2019 to present.

Methodology: We prove the existence, nonexistence and multiplicity of positive solutions by the results of fixed point index.

Results: We give new conclusions of existence, nonexistence and multiplicity of positive solutions for the system.

Conclusion: We prove the existence, nonexistence and multiplicity of positive solutions to the $n$-dimensional elliptic system
\[
\begin{align*}
\Delta u(x) + \Lambda a(|x|)f(u(x)) &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
and give new conclusions.

Keywords: $n$-dimensional elliptic system; fixed point index; existence; nonexistence; multiplicity.

2010 Mathematics Subject Classification: 53C25; 83C05; 57N16.

1 Introduction

This paper considers the existence, nonexistence and multiplicity of positive solutions to the following $n$-dimensional elliptic system
\[
\begin{align*}
\Delta u(x) + \Lambda a(|x|)f(u(x)) &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\] (1.1)
where $\Omega = \{x \in \mathbb{R}^n : R_1 < |x| < R_2\}, R_1 > R_2 > 0, n \geq 2, a(|x|)$ is allowed to change sign on $[R_1, R_2]$ and
\[
x = (x_1, x_2, ..., x_n)^T, \\
u(x) = (u_1(x), ..., u_k(x), ..., u_n(x))^T, \\
\Delta u(x) = (\Delta u_1(x), ..., \Delta u_k(x), ..., \Delta u_n(x))^T, \\
\Lambda = \text{diag}[\lambda_1, ..., \lambda_k, ..., \lambda_n], \\
a(|x|) = \text{diag}[a_1(|x|), ..., a_k(|x|), ..., a_n(|x|)], \\
f(x) = (f_1(x), ..., f_k(x), ..., f_n(x))^T,
\]
where $f_k(x)$ can be understood as $f_k(x_1, x_2, ..., x_n), k \in \{1, 2, ..., n\}$.

Then, system (1.1) means that $k \in \{1, 2, ..., n\}$
\[
\begin{align*}
\Delta u_k(x) + \lambda_k a_k(|x|)f_k(u(x)) &= 0 \quad \text{in } \Omega, \\
u_k &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\] (1.2)

Let $J = [0, 1], R_+ = [0, +\infty), R = [-\infty, +\infty), R_+^n = R_+ \times R_+ \times \cdots \times R_+.$

Referring to [1], by a positive solution of (1.1) is meant a solution $u \in C^2(\Omega)$ with $u \geq 0$ in $\Omega$. By the strong maximum principle, it follows that $u > 0$ in $\Omega$.

In recent years, people have extensively studied the positive solutions of elliptic system see [2-6,7,8-24].

In [17], under mixed Dirichlet boundary conditions and Newman boundary conditions, Dunninger and Wang studied the existence, nonexistence and multiplicity of positive solutions for (1.3) by
applying fixed point index. Under Dirichlet boundary conditions, Lee studied the existence and multiplicity of positive solutions for (1.3) by applying Fixed point theorem of cone expansion and compression of norm type in [19]; Dunninger and Wang studied the existence and multiplicity of positive solutions for (1.3) by Fixed point theorem in [20].

\[
\begin{align*}
\Delta u + \lambda_1 a_1(|x|) f(u, v) &= 0, \\
\Delta v + \lambda_2 a_1(|x|) g(u, v) &= 0,
\end{align*}
\]

(1.3)

Under Dirichlet boundary conditions, Ma studied the existence of positive radial solutions of (1.4) by making use of the fixed point theorem in [22]; By applying the classical fixed point index theorem, Infante and Pietramala showed the existence, localization, nonexistence and multiplicity of positive solutions of (1.4) in [5]; Do, Lorca, Sanchez and Ubilla by using a fixed point theorem due to M.A. Krasnoselskii, the sub- and super- solutions method and fixed point index theory studied existence, non-existence and multiplicity of positive solutions for problem (1.4) in [6]; Precup by using the version Krasnoselskii’s cone fixed point theorem studied the existence, localization and multiplicity of positive radial solutions to (1.4) in [24].

\[
\begin{align*}
\Delta u + a_1(|x|) f(u, v) &= 0, \\
\Delta v + a_1(|x|) g(u, v) &= 0,
\end{align*}
\]

(1.4)

However, there are relatively few studies on \( n \)-dimensional elliptic systems. In [1], Feng and Li studied the existence and multiplicity of positive solutions for (1.1) by Fixed point theorem of cone expansion and compression of norm type. In [25], Fotouhi, Shahgholian and Weiss by the epiperimetric inequality approach studied the following system

\[
\Delta u = \lambda_+(x)|u^+|^{q-1} u^+ - \lambda_-(x)|u^-|^{q-1} u^-.
\]

In spired by the works above, this paper is devoted to studying the nonexistence and to obtaining new results on the existence and multiplicity of system (1.1).

By letting \( r = |x|, s = - \int_{m}^{R} (1/t^{n-1}) dt, m = - \int_{0}^{R} (1/t^{n-1}) dt, v_k(s) = u_k(r(s)), t = (m - s)/m, w_k(t) = v_k(s) \), the system (1.2) transforms into the following system

\[
\begin{align*}
w_k''(t) + \lambda k h_k(t) f_k(w(t)) &= 0, \\
w_k(0) &= w_k(1) = 0.
\end{align*}
\]

(1.5)

where \( k \in \{1, 2, ..., n\} \), \( h_k(t) = m^2 r^{2(n-1)} (m(1-t)) a_i(r(m(1-t))) \). Referring to [1] for the specific process. Thus the study of (1.1) becomes the study of (1.5).

The following conditions will be assumed throughout this paper:

\[
\begin{align*}
(H_1) & \lambda_k > 0, \ k \in \{1, 2, ..., n\}; \\
(H_2) & h_k : [0, 1] \rightarrow \mathbb{R} \text{ is continuous and does not vanish identically on any subinterval } [0, 1], k \in \{1, 2, ..., n\}. \\
(H_3) & f_k : \mathbb{R}^n_+ \rightarrow \mathbb{R}_+ \text{ is continuous, } k \in \{1, 2, ..., n\}. \\
(H_4) & f_k(w_1) \leq f_k(w_2) \text{ for } 0 \leq w_1 \leq w_2. \\
(H_5) & \lim_{m_{1,2} \leq j \leq w_{1,2} \rightarrow \infty} \frac{f_k(w_1)}{w_k} = \infty.
\end{align*}
\]

The purpose of this paper is to study the system (1.1) and prove the following theorems.

**Theorem 1.1.** Suppose that (H1), (H2), (H3), (H4), (H5) hold, then there exists \( \lambda^*_k > 0 \) such that (1.1) has at least two positive solutions for \( 0 < \lambda_k < \lambda^*_k \), at least one positive solution for \( \lambda_k = \lambda^*_k \) and no positive solution for \( \lambda_k > \lambda^*_k \).
2 Preliminary Lemmas

In this section, we give some lemmas that required.

Lemma 2.1. Let $X$ be a Banach space and $K$ a cone in $X$. For $r > 0$, define $K_r = \{ x \in K | \|x\| < r \}$. Assume that $T : K_r \to K$ is a compact map such that $Tx \neq x$ for $x \in \partial K_r$.

(i) If $\|x\| \leq \|Tx\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 0$.

(ii) If $\|x\| \geq \|Tx\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 1$.

Lemma 2.2. Let $X$ be a Banach space and $K$ a cone in $X$ and $\Omega$ a bounded open set in $X$. Let $0 \in \Omega$ and $T : K \cap \Omega \to K$ be condensing. Suppose that $Tx \neq \mu x$, for all $x \in K \cap \partial \Omega$ and all $\mu \geq 1$.

Then, $i(T, K \cap \Omega, K) = 1$.

3 Existence and Nonexistence

It is well know that the system (1.5) has a solution $w = (w_1, ..., w_k, ..., w_n)$ given by

$$w_k(t) = \lambda_k \int_0^1 G(t, s)h_k(s)f_k(w(s))ds, \quad t \in J,$$

where $G(t, s)$ is the Green’s function and the exact expression is

$$G(t, s) = \begin{cases} t(1-s) & \text{for } 0 \leq t \leq s \leq 1, \\ s(1-t) & \text{for } 0 \leq s \leq t \leq 1. \end{cases}$$

We can easily see that for any $t \in J$,

$$G(t, s) \leq G(s, s) \leq \frac{1}{4}.$$  \hspace{1cm} (3.1)

for any $t \in [\frac{1}{4}, \frac{3}{4}]$.

$$G(t, s) \geq \frac{1}{4}G(s, s).$$  \hspace{1cm} (3.2)

Let $X = \prod_{k=1}^n C[0,1], ||x||_\infty = \max_{t \in J} |x(t)|$, and for any $x = (x_1, x_2, ..., x_n)^T \in X, k \in \{1, 2, ..., n\}$,

$$||x|| = \sum_{k=1}^n ||x_k||_\infty.$$

Then $(X, || \cdot ||)$ is a real Banach space.

Define a cone $K$ by

$$K = \{ w = (w_1, ..., w_k, ..., w_n) \in X : w_k(t) \geq 0, \quad t \in [0,1], k \in \{1, 2, ..., n\}, \quad \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \sum_{k=1}^n w_k(t) \geq \frac{1}{4}||w|| \}. $$

Let $C$ be the cone defined by

$$C = \{ w = (w_1, ..., w_k, ..., w_n) \in X : w_k(t) \geq 0, \quad t \in J, k \in \{1, 2, ..., n\} \}.$$

We define the map $T^A : \mathbb{R}_+^n \to \mathbb{R}_+$ with components $(T_1^{\lambda_1}, ..., T_k^{\lambda_k}, ..., T_n^{\lambda_n})$. Here we understand $T^A w = (T_1^{\lambda_1} w, ..., T_k^{\lambda_k} w, ..., T_n^{\lambda_n} w)$, which

$$T_k^{\lambda_k} w(t) = \lambda_k \int_0^1 G(t, s)h_k(s)f_k(w(s))ds, t \in J, k \in \{1, 2, ..., n\}. $$
Referring to [1], we can obtain the existence of a positive solution of system (1.3) is equivalent to the fixed-point system

\[ T^\lambda w = w. \]

Set

\[ K_r = \{ w \in K ||w|| < r \}, \]
\[ \bar{K}_r = \{ w \in K ||w|| \leq r \}. \]

**Lemma 3.1.** \( T^\lambda : X \rightarrow X \) is completely continuous and \( T^\lambda(C) \subset K \).

**Proof:** To prove \( T^\lambda(C) \subset K \), choose \( w \in C \). Then for \( t \in \left[ \frac{1}{4} \right. \frac{3}{4} \]

\[ T^\lambda w(t) = (T^\lambda w_1, ..., T^\lambda w_n) \]
\[ \geq \left( \frac{1}{4} \lambda_1 \int_0^1 G(s, s)h_1(s)f_1(w(s))ds, ..., \frac{1}{4} \lambda_n \int_0^1 G(s, s)h_n(s)f_n(w(s))ds \right) \]
\[ := \frac{1}{4} T^\lambda w(s), \]

for all \( s \in J \), and so

\[ \min_{t \in \left[ \frac{1}{4} \frac{3}{4} \right]} \left| \sum_{k=1}^n T^\lambda w_k \right| \geq \frac{1}{4} ||T^\lambda w||, \]

i.e., \( T^\lambda w \subset K \); Hence, \( T^\lambda(C) \subset K \).

Referring to the proof Lemma 2.4 in [26], we can find the complete continous of \( T^\lambda \) is obvious. Then we complete our proof of Lemma 3.1.

**Theorem 3.2.** Suppose that \( (H_1), (H_2), (H_3), (H_4), (H_5) \) hold, then for \( \lambda_k, k = \{1, 2, ..., n\} \) sufficiently small, \((1.1) \) has at least one positive solution, whereas for \( \lambda_k, k = \{1, 2, ..., n\} \) sufficiently large, \((1.1) \) has no positive solution.

**Proof:** If \( q > 0 \), then it follows from \((H_2)\) and \((H_3)\) that

\[ \beta(q) = \frac{1}{4} \sum_{k=1}^n \max_{w \in K, ||w|| = q} \left( \int_0^1 G(t, s)h_k(s)f_k(w(s))ds \right) > 0. \]

For any number \( 0 < r_1 \), let \( \delta = r_1 / \beta(r_1) \).

Then for \( \lambda_k < \delta \) and \( w_k \in \partial K_{r_1}, k = \{1, 2, ..., n\} \), we have

\[ ||T_{\lambda_k}^\lambda w(t)||_{\infty} = \max_{t \in J} \left| \int_0^1 G(t, s)h_k(s)f_k(w(s))ds \right| \]
\[ \leq \frac{1}{4} \lambda_k \int_0^1 h_k(s)f_k(w(s))ds \]
\[ \leq \frac{1}{4} \delta \int_0^1 h_k(s)f_k(w(s))ds, \]

\[ ||T^\lambda w(t)|| = \sum_{k=1}^n ||T_{\lambda_k}^\lambda w(t)||_{\infty} \]
\[ \leq \sum_{k=1}^n \left| \int_0^1 h_k(s)f_k(w(s))ds \right| \]
\[ \leq \delta \beta(r_1) \]
\[ = r_1 \]
\[ = ||w||. \]
By Lemma 2.1, we can obtain $i(T^k, K_{r_1}, K) = 1$.

From $(H_5)$ we can get there is an $H > 0$ such that $f_k(w) \geq \eta_k \sum_{k=1}^{n} w_k$ for $w \geq H$, where $\eta_k$ is chosen so that $\sum_{k=1}^{n} \frac{1}{16} \int_{\frac{1}{4}}^{\frac{3}{4}} G(s, s)h_k(s)\eta_k ds > 1, k \in \{1, 2, ..., n\}$.

Let $r_2 \geq 4H$, for $w_k \in \partial K_{r_2}, k \in \{1, 2, ..., n\}$, we have

$$
\|T^k w(t)\| = \max_{t \in J} |\lambda_k \int_{0}^{1} G(t, s)h_k(s)f_k(w(s))ds|
\geq \frac{1}{4} \lambda_k \int_{0}^{1} G(s, s)h_k(s)f_k(w(s))ds
\geq \frac{1}{4} \lambda_k \int_{0}^{1} G(s, s)h_k(s)\eta_k \sum_{k=1}^{n} w_k(s)ds
\geq \frac{1}{4} \lambda_k \int_{\frac{1}{4}}^{\frac{3}{4}} G(s, s)h_k(s)\eta_k \sum_{k=1}^{n} w_k(s)ds
> \frac{1}{4} \lambda_k \int_{\frac{1}{4}}^{\frac{3}{4}} G(s, s)h_k(s)\eta_k \|w\|ds
= \frac{1}{16} \lambda_k \int_{\frac{1}{4}}^{\frac{3}{4}} G(s, s)h_k(s)\eta_k \|w\|ds,
$$

$$
\|T^k w(t)\| = \sum_{k=1}^{n} \|T^k w(t)\| = \sum_{k=1}^{n} \left( \frac{1}{16} \lambda_k \int_{\frac{1}{4}}^{\frac{3}{4}} G(s, s)h_k(s)\eta_k ds \right) \|w\|
\geq \frac{1}{16} \lambda_k \int_{\frac{1}{4}}^{\frac{3}{4}} G(s, s)h_k(s)\eta_k ds \|w\|
\geq \|w\|.
$$

By Lemma 2.1, we can obtain $i(T^k, K_{r_2}, K) = 0$.

Since we can adjust $r_1, r_2$ so that $r_1 < r_2$, it follows from the additivity of the fixed point index that $i(T^k, K_{r_2} \setminus K_{r_1}, K) = -1$. Thus, $K$ has a fixed point in $K_{r_2} \setminus K_{r_1}$ which is the desired positive solution of (1.1).

To prove the nonexistence part, because of $(H_5)$ we can get existence of a constant $c_k$ such that $f_k(w) \geq c_k \sum_{k=1}^{n} w_k$ for $w_k \geq 0$, where $c_k, k \in \{1, 2, ..., n\}$ is chosen so that

$$
\left( \frac{1}{16} c_k \lambda_1 \int_{\frac{1}{4}}^{\frac{3}{4}} G(s, s)h_1(s)ds, ..., \frac{1}{16} c_k \lambda_n \int_{\frac{1}{4}}^{\frac{3}{4}} G(s, s)h_n(s)ds \right) > 1.
$$
Let \( w \in X \) be a positive solution of (1.1). By Lemma 3.1, \( w \in K \), for all \( t \in \left[ \frac{1}{4}, \frac{3}{4} \right] \), we have

\[
T_k^w w = \lambda_k \int_0^1 G(t, s) h_k(s) f_k(w(s)) ds \\
\geq \frac{1}{4} \lambda_k \int_0^1 G(s, s) h_k(s) f_k(w(s)) ds \\
\geq \frac{1}{4} \lambda_k \int_0^1 G(s, s) h_k(s) c_k \sum_{k=1}^n w_k(s) ds \\
\geq \frac{1}{4} \lambda_k \int_\frac{3}{4}^1 G(s, s) h_k(s) c_k \sum_{k=1}^n w_k(s) ds \\
\geq \frac{1}{4} \lambda_k \int_{\frac{1}{4}}^\frac{3}{4} G(s, s) h_k(s) c_k \left( \frac{1}{4} \| w \| ds \\
= \frac{1}{16} \lambda_k \int_{\frac{1}{4}}^\frac{3}{4} G(s, s) h_k(s) c_k ds \| w \| \\
\geq (16 c_1 \lambda_1 \int_{\frac{1}{4}}^\frac{3}{4} G(s, s) h_1(s) ds \| w \|, ..., 16 c_n \lambda_n \int_{\frac{1}{4}}^\frac{3}{4} G(s, s) h_n(s) ds \| w \|) \\
= \| w \|
\]

for \( \lambda_k, k \in \{1, 2, ..., n\} \) is large enough, which is a contradiction, then we complete our proof.

4 Multiplicity

**Lemma 4.1.** For \((H_5)\), then there is a constant \( b_f > 0 \) such that \( \| w \| \leq b_f \) for all positive solutions of (1.1), where \( \lambda_k, k \in \{1, 2, ..., n\} \) belongs to a compact subset \( I \) of \((0, \infty)\).

**Proof:** Suppose there is a sequence \( \{ w_n \} \) of positive solution of, with corresponding \( \lambda_{k,n}, k \in \{1, 2, ..., n\} \) belonging to a compact subset of \((0, \infty)\), such that \( \lim_{n \to \infty} \| w_n \| = \infty \). By Lemma 3.1, \( w_n \in K \). As before, there is an \( H > 0 \) such that \( f_k(w) \geq \eta_k \sum_{k=1}^n w_k \) for \( w \geq H \), where \( \eta_k \) is choosen so that

\[
\left( \frac{1}{16} \lambda_1 \int_{\frac{1}{4}}^\frac{3}{4} G(s, s) h_1(s) \eta_1 ds, ..., \frac{1}{16} \lambda_n \int_{\frac{1}{4}}^\frac{3}{4} G(s, s) h_n(s) \eta_n ds \right)
\]
1, k ∈ \{1, 2, ..., n\}.

\[ T^{\lambda_k}_k \mathbf{w} = \lambda_k \int_0^1 G(t, s)h_k(s)f_k(\mathbf{w}(s))ds \]

\[ \geq \frac{1}{4} \lambda_k \int_0^1 G(s, s)h_k(s)ds \]

\[ \geq \frac{1}{4} \lambda_k \int_0^1 G(s, s)h_k(s)(n)w_k(s))ds \]

\[ \geq \frac{1}{4} \lambda_k \int_0^1 G(s, s)h_k(s)\eta \lambda_k \int_0^1 G(s, s)h_k(s)\eta \mathbf{w}(s)ds \]

\[ = \frac{1}{16} \lambda_k \int_0^1 G(s, s)h_k(s)\eta \mathbf{w}(s)ds \]

\[ \|\mathbf{w}(t)\| \geq \mathbf{w}(t) \]

\[ = T^\lambda \mathbf{w} \]

\[ = (T^{\lambda_1}_1 \mathbf{w}, ..., T^{\lambda_k}_k \mathbf{w}, ..., T^{\lambda_n}_n \mathbf{w}) \]

\[ \geq \left( \frac{1}{16} \lambda_1 \right) \int_0^1 G(s, s)h_1(s)\eta ds \|\mathbf{w}\|, ... \]

\[ \geq \left( \frac{1}{16} \lambda_k \right) \int_0^1 G(s, s)h_k(s)\eta ds \|\mathbf{w}\| \]

\[ = \left( \frac{1}{16} \lambda \right) \int_0^1 G(s, s)h(s)\eta ds, ..., \lambda \int_0^1 G(s, s)h(s)\eta ds \] (4.1)

A lower solution \( \mathbf{w} \geq 0 \) is defined similarly by reversing the inequalities in (4.1).

**Lemma 4.2.** If there exists an upper solution \( \mathbf{w} \) of (1.1), then there is a positive solution \( \mathbf{w} \) of (1.1) with \( 0 \leq \mathbf{w} \leq \mathbf{w} \).

**Proof:** By taking into account the monotonicity conditions \((H_k)\), and noting that \( \mathbf{0} \) is a lower solution of (1.1), it follows that the usual monotone iteration scheme applies and this completes the proof.

Now let \( \Gamma \) denote the set of \( \lambda_k > 0 \) such that a positive solution of (1.1) exists, and set \( \lambda_k = \sup \Gamma \), by Theorem 3.2, \( \Gamma \) is nonempty and bounded, and thus \( 0 < \lambda_k < \infty \), we claim that \( \lambda_k \in \Gamma \), to see this, let \( \lambda_{k,n} \to \lambda_k \), where \( \lambda_{k,n} \in \Gamma, k \in \{1, 2, ..., n\} \).

Since the \( \lambda_{k,n} \) are bounded, Lemma 4.1 implies that the corresponding solutions \( \mathbf{w}_n \) are bounded, by the compactness of the integral operator \( T^\lambda \), it easily follows that \( \lambda_k \in \Gamma \), let \( \mathbf{w}^* \) be a positive solution of (1.1) corresponding to \( \lambda_k^* \), k \in \{1, 2, ..., n\}.
Lemma 4.3. Let $0 < \lambda_k < \lambda_k^*$, $k \in \{1, 2, ..., n\}$. Then there exists $\epsilon^* > 0$ such that $w^* + \epsilon, 0 \leq \epsilon \leq \epsilon^*$ is an upper solution of (1.1).

Proof: Since $w^* \geq 0$, there is a constant $c$ such that $f(w^*) \geq c > 0$ for all $t \in [0, 1]$. By uniform continuity, there is an $\epsilon^*$ such that

$$|f_k(w^* + \epsilon) - f_k(w^*)| < c(\lambda_k^* - \lambda_k)/\lambda_k,$$

for all $t \in J, 0 \leq \epsilon \leq \epsilon^*, k \in \{1, 2, ..., n\}$. Note $F_k^0 w^* = \lambda_k^* \int_0^1 G(t, s) h_k(s) f_k(w^*(s)), F_k^1 w^* = \lambda_k^* \int_0^1 G(t, s) h_k(s) f_k(w^*(s) + \epsilon) - f_k(w^*(s)), F_k^2 w^* = (\lambda_k^* - \lambda_k) \int_0^1 G(t, s) h_k(s) f_k(w^*(s)), k \in \{1, 2, ..., n\}.$

$$w^*(t) + \epsilon \geq (F_1^0 w^*, ..., F_n^0 w^*) = (F_1^2 w^*, ..., F_n^2 w^*) - (F_1^2 w^*, ..., F_n^2 w^*) + (F_1^3 w^*, ..., F_n^3 w^*, ..., F_n^2 w^*),$$

where the first inequality is strict if $\epsilon > 0$. A similar calculation is valid for $w^* + \epsilon$, thus this completes the proof.

Proof of Theorem 1.1. Let $0 < \lambda_k < \lambda_k^*, k \in \{1, 2, ..., n\}$. Since $w^*$ is an upper solution of (1.1), Lemma 4.2 implies the existence of a positive solution $w^0$ of (1.1) with $0 < w^0 \leq w^*$. Thus for $0 < \lambda_k < \lambda_k^*, i = 1, 2, ..., n$ a positive solution exists, whereas for $\lambda_k > \lambda_k^*, k \in \{1, 2, ..., n\}$ a positive solution does not exist. We next establish the existence of a second positive solution of (1.1) for $0 < \lambda_k < \lambda_k^*, k \in \{1, 2, ..., n\}$. Consider

$$\Omega = \{w \in X; -\epsilon < w < w^*(s) + \epsilon, t \in J\},$$

where $\epsilon > 0$ is given in Lemma 4.3. Then $\Omega$ is bounded and open in $X, 0 \in \Omega$, and $T: K \subset \Omega \rightarrow K$ is condensing since it is compact. Moreover, $w^0 \in \Omega$ for $0 < \lambda_k \leq \lambda_k^*, k \in \{1, 2, ..., n\}.$

Let $w \in K \cap \partial \Omega$. Then there is a $t_0$ such that $w(t_0) = w^*(t_0) + \epsilon$ and taking into account, note $F_k w = \lambda_k \int_0^1 G(t, s) h_k(s) f_k(w(s) + \epsilon), k \in \{1, 2, ..., n\},$

we have

$$T^\mu w(t_0) = (T_1^\mu w(t_0), ..., T_n^\mu w(t_0), ..., T_n^\mu w(t_0))$$

$$\leq (F_1 w, ..., F_n w)$$

$$= w(t_0) + \epsilon$$

$$\leq \mu w(t_0),$$

for all $\mu \geq 1$. By Lemma 2.2, $i(T, K \cap \Omega, K) = 1.$

Next let $r = max\{b_2 + 1, r_2, ||w^* + \epsilon||\}$, where $b_2$ is given in Lemma 4.1, and $r_2$ is given in the proof of Theorem 3.1. Lemma 4.1 implies that $T^\mu w \neq w$ for $w \in \partial K_r$. Furthermore, if $w \in \partial K_r$, then, as in the proof of Theorem 3.1, we see that $||T^\mu w|| \geq ||w||$. Consequently, Lemma 2.1 implies $i(T, K_r \setminus (\bar{K} \cap \Omega), K) = 0$, and by the additivity of the fixed point index we get $i(T, K_r \setminus (\bar{K} \cap \Omega), K) = -1$.

Thus, $T^\mu$ has a fixed point in $K_r \setminus (\bar{K} \cap \Omega)$, which establishes the existence of a second positive solution. And we complete our proof.
5 Conclusions

Suppose that \((H_1), (H_2), (H_3), (H_4), (H_5)\) hold, then there exists \(\lambda_0 > 0\) such that (1.1) has at least two positive solutions for \(0 < \lambda_k < \lambda_0\), at least one positive solution for \(\lambda_k = \lambda_0\) and no positive solution for \(\lambda_k > \lambda_0\).

Acknowledgement

Thanks to my teacher's help in research, modification and so on, this paper can be completed. At the same time, I would like to thank Beijing Information Science & Technology University for providing the good learning conditions.

Competing Interests

Author has declared that no competing interests exist.

References


© 2021 Shen; This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution and reproduction in any medium, provided the original work is properly cited.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar).
http://www.sdiarticle4.com/review-history/72817