Existence and Nonexistence of Nontrivial Doubly Periodic Solutions of Nonlinear Telegraph Equations

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Author’s contribution

The calculation, proof and writing of this article are completed by the author independently.

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Abstract

Aims/ Objectives: We discuss the existence and nonexistence of nontrivial nonnegative doubly periodic solutions for nonlinear telegraph equations

$$u_{tt} - u_{xx} + cu_t + a(t, x)u = \lambda f(t, x, u),$$

where $c > 0$ is a constant, $\lambda > 0$ is a positive parameter, $a \in C(\mathbb{R}^2, \mathbb{R}^+)$, $f \in C(\mathbb{R}^2 \times \mathbb{R}^+, \mathbb{R}^+)$, and $a, f$ are $2\pi$-periodic in $t$ and $x$. The proof is based on a known fixed point theorem due to Schauder. In previous articles, a single telegraph equation or telegraph system have been widely studied, but there is relatively little research on nonlinear telegraph equations with a parameter and the nonlinearities are nonnegative. We would like do some research on this topic. We give new conclusions on the existence and nonexistence of nontrivial nonnegative doubly periodic solutions for nonlinear telegraph equations under sublinear assumptions.

Study Design: Study on the existence and nonexistence of nontrivial nonnegative doubly periodic solutions.

Place and Duration of Study: School of Applied Science, Beijing Information Science & Technology University, September 2020 to present.

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Methodology: We prove the existence and nonexistence of nontrivial nonnegative doubly periodic solutions by the results of Schauder’s fixed point theorem.

Results: We give new conclusions of existence and nonexistence of nontrivial nonnegative doubly periodic solutions for the equations.

Conclusion: We prove the existence and nonexistence of nontrivial nonnegative doubly periodic solutions for the nonlinear telegraph equation
\[ u_{tt} - u_{xx} + cu_t + a(t,x)u = \lambda f(t,x,u), \]
and give new conclusions.

Keywords: Doubly periodic solution; telegraph equation; fixed point theorem; existence and nonexistence.

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1 Introduction

In this paper, we are concerned with the existence and nonexistence of nonnegative doubly periodic solutions for the nonlinear telegraph equation
\[ u_{tt} - u_{xx} + cu_t + a(t,x)u = \lambda f(t,x,u) \quad (t,x) \in \mathbb{R}^2, \quad (1.1) \]
and they are doubly periodic in the following sense
\[ u(t + 2\pi, x) = u(t, x + 2\pi) = u(t, x) \quad (t,x) \in \mathbb{R}^2, \]
where \( c \) is a constant and \( c > 0 \), \( \lambda \) is a positive parameter, \( a \in C(\mathbb{R}^2, \mathbb{R}^+), f \in C(\mathbb{R}^2 \times \mathbb{R}^+, \mathbb{R}^+), \) and \( a \) and \( f \) are \( 2\pi \)-periodic in \( t \) and \( x \).

The results concerning the existence and multiplicity of nontrivial doubly periodic solutions for a single telegraph equation or telegraph system, see [1-20] and the references therein. By using weak force conditions, Wang [20] constructed some existence results for the following periodic boundary value problems:
\[
\begin{aligned}
& u_{tt} - u_{xx} + c_1u_t + a_{11}(t,x)u + a_{12}(t,x)v = f_1(t,x,u,v) + \chi_1(t,x), \\
& v_{tt} - v_{xx} + c_2v_t + a_{21}(t,x)u + a_{22}(t,x)v = f_2(t,x,u,v) + \chi_2(t,x).
\end{aligned}
\quad (1.2)
\]
The proof is based on Schauder’s fixed point theorem. Wang [8] pays attention to the existence and multiplicity of double periodic solutions for the nonlinear telegraph system
\[
\begin{aligned}
& u_{tt} - u_{xx} + c_1u_t + a_{11}(t,x)u + a_{12}(t,x)v = \lambda f_1(t,x,u,v), \\
& v_{tt} - v_{xx} + c_2v_t + a_{21}(t,x)u + a_{22}(t,x)v = \lambda f_2(t,x,u,v),
\end{aligned}
\quad (1.3)
\]
where \( f_i(t,x,u,v) \in C(\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \) and is bounded below. The proof is based on a well known fixed theorem in a cone. Using fixed-point theorem of a cone mapping, Wang [14] pays attention to the multiplicity of positive doubly periodic solutions of (1.1), where \( f(t,x,u) \) may change sign and is singular at \( u = 0 \).

If the nonlinearities are nonnegative, we show that the problem (1.1) admits existence and nonexistence of doubly periodic solutions. In this paper, we will use of the Schauder’s fixed point theorem to prove that the problem (1.1) admits one nontrivial nonnegative solution for small \( \lambda > 0 \) if one of \( \lim_{u \to 0^+} \frac{f(t,x,u)}{u} \) is infinity. In addition, all \( \lim_{u \to \infty} \frac{f(t,x,u)}{u} \) is zero, we show that the problem (1.1) admits a nontrivial solution for all \( \lambda > 0 \).
We also provide a result that there is nonexistence of doubly periodic solution.

The paper is organized as follows: In Section 2, we make some preliminaries; In section 3, we prove the existence and nonexistence of nontrivial nonnegative doubly periodic solutions for problem (1.1).

2 Preliminaries

Let $\mathbb{T}^2$ be the torus defined as

$$\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z}).$$

Doubly $2\pi$-periodic functions will be defined on $\mathbb{T}^2$. We use the notations $L^p(\mathbb{T}^2)$, $C(\mathbb{T}^2)$, $C^\omega(\mathbb{T}^2)$, $D(\mathbb{T}^2) = C^\omega(\mathbb{T}^2)$, etc.

to denote the spaces of doubly periodic functions with the indicated degree of regularity. The space $D'(\mathbb{T}^2)$ denotes the space of distributions on $\mathbb{T}^2$.

By a doubly periodic solution of (1.1) we mean that $u \in L^1(\mathbb{T}^2)$ satisfies (1.1) in the distribution sense, that is

$$\int_{\mathbb{T}^2} u(\phi_{tt} - \phi_{xx} - c\phi_t + a(t,x)\phi) dtdx = \int_{\mathbb{T}^2} \lambda f(t,x,u(t,x))\phi dtdx, \quad \forall \phi \in D(\mathbb{T}^2)$$

with $f(.,u(.,.)) \in L^1(\mathbb{T}^2)$.

In this section, we consider the positivity estimation for the linear equation

$$u_{tt} - u_{xx} + cu_t + a(t,x)u = h(t,x), \quad \text{in } D'(\mathbb{T}^2).$$

(2.2)

Let $c$ is a constant and $c > 0$. The linear differential operator is defined as

$$Lu = u_{tt} - u_{xx} + cu_t,$$

acting on functions on the torus $\mathbb{T}^2$, $u : \mathbb{T}^2 \to \mathbb{R}$. We define the formal adjoint operator

$$L^*u = u_{tt} - u_{xx} - cu_t.$$

Given $\sigma \in \mathbb{R}$ and $h \in L^1(\mathbb{T}^2)$, we consider the linear problem

$$Lu - \sigma u = h(t,x), \quad \text{in } D'(\mathbb{T}^2).$$

(2.3)

A solution of (2.3) is a function $u \in L^1(\mathbb{T}^2)$ satisfying

$$\int_{\mathbb{T}^2} u(L^*\phi - \sigma\phi) dtdx = \int_{\mathbb{T}^2} h\phi dtdx, \quad \forall \phi \in D(\mathbb{T}^2).$$

Let $L_\sigma$ be the differential operator

$$L_\sigma u = Lu - \sigma u = u_{tt} - u_{xx} + cu_t - \sigma u$$

acting on functions on $\mathbb{T}^2$. By [3,4], if $\sigma < 0$, then $L_\sigma$ has the resolvent $R_\sigma$

$$R_\sigma : L^1(\mathbb{T}^2) \to C(\mathbb{T}^2), \quad h \mapsto u,$$

where $u$ is the unique solution of Eq. (2.3), and the restriction of $R_\sigma$ on $L^p(\mathbb{T}^2)(1 < p < \infty)$ or $C(\mathbb{R})$ is compact. In particular, $R_\sigma : C(\mathbb{T}^2) \to C(\mathbb{T}^2)$ is a completely continuous operator.
For $\sigma = c^2/4$, the Green’s function of the differential operator $L_\sigma$ can be explicitly expressed, which has been obtained in [3]. We denote it by $G(t, x)$. By [Lemma 5.1 in [3]], $G$ is doubly $2\pi$-periodic, and given $h \in L^1(\mathbb{T}^2)$, the unique solution of (2.1) can be represented by convolution product
\[
u(t, x) = (R_\mu h)(t, x) = \int_{\mathbb{T}^2} G(t - s, x - y) h(s, y) ds dy.
\]
(2.4)
The expression of $G(t, s)$ will be given in the following.

Let $D = \mathbb{R}^2 \setminus C$, where $C$ is the family of lines
\[x \pm t = 2k\pi, \quad k \in \mathbb{Z}.
\]Let $D_{ij}$ denote the connected component of $D$ with center at $(i\pi, j\pi)$, where $i + j$ is an odd number.

By periodicity, the value of $G$ on $D_{10}$ and $D_{01}$ completely determines the value of $G$ on the whole set $D$. In $D_{10}$ and $D_{01}, G(t, x)$ is explicitly given by
\[
G(t, x) = \begin{cases}
\gamma_{10} e^{-ct/2}, & (t, x) \in D_{10}, \\
\gamma_{01} e^{-ct/2}, & (t, x) \in D_{01},
\end{cases}
\]
(2.5)
where
\[
\gamma_{10} = \frac{1 + e^{-c\pi}}{2 (1 - e^{-c\pi})^2}, \quad \gamma_{01} = \frac{1 + e^{-c\pi}}{(1 - e^{-c\pi})^2}.
\]
See [3], Lemma 5.2.

From (2.5), we have
\[
\underline{G} := \text{ess inf } G(t, x) = \frac{e^{-3\pi/2}}{(1 - e^{-c\pi})^2},
\]
(2.6)
\[
\overline{G} := \text{ess sup } G(t, x) = \frac{1 + e^{-c\pi}}{2 (1 - e^{-c\pi})^2}.
\]
(2.7)
We will use the following two lemmas to simplify the proofs of our existence theorems. More importantly, the monotonicity assumptions of the nonlinearities can be relaxed.

Let $\delta > 0, f : \mathbb{T}^2 \times \mathbb{R}^+ \to \mathbb{R}^+$ be continuous. We define two new functions: $f^{\text{min}}(t, x, z) : \mathbb{T}^2 \times [0, \delta) \to \mathbb{R}^+$ and $f^{\text{max}}(t, x, z) : \mathbb{T}^2 \times \mathbb{R}^+ \to \mathbb{R}^+$ by
\[
f^{\text{min}}(t, x, z) = \min \{f(t, x, u) : u \in \mathbb{R}^+, z \leq u(t, x) \leq \delta \text{ and } (t, x) \in \mathbb{T}^2\}
\]
and
\[
f^{\text{max}}(t, x, z) = \max \{f(t, x, u) : u \in \mathbb{R}^+, u(t, x) \leq z \text{ and } (t, x) \in \mathbb{T}^2\}.
\]
It is clear that both $f^{\text{min}}$ and $f^{\text{max}}$ are nondecreasing.

The proof of the following two lemmas can be found in Hai [21].

**Lemma 2.1.** ([21]) If
\[
f(t, x, u) > 0 \quad \text{for } 0 < u, \quad (t, x) \in \mathbb{T}^2,
\]
and
\[
\lim_{u \to -0^+} \frac{f(t, x, u)}{u} = \infty, \quad u \in \mathbb{R}^+, \quad (t, x) \in \mathbb{T}^2,
\]
then
\[
\lim_{z \to 0^+} \frac{f^{\text{min}}(t, x, z)}{z} = \infty.
\]
Lemma 2.2. ([22,21]) Let $u \in \mathbb{R}^+$ and $(t, x) \in T^2$, then assume $\lim_{u \to 0^+} \frac{f(t, x, u)}{u}$ and $\lim_{u \to 0^+} \frac{f(t, x, u)}{u}$ exist (can be infinity). Then

$$\lim_{t \to 0^+} \frac{f_{\max}(t, x, z)}{z} = \lim_{u \to 0^+} \frac{f(t, x, u)}{u}$$

and

$$\lim_{t \to \infty} \frac{f_{\max}(t, x, z)}{z} = \lim_{u \to \infty} \frac{f(t, x, u)}{u}.$$ 

To prove our results, we need the following fixed-point theorem of Schauder.

Lemma 2.3. ([23] (Schauder)) Let $X$ be a Banach space and $D \subseteq X$ be a bounded, convex and closed subset. Assume that $T : D \to D$ is completely continuous, then $T$ has a fixed point in $D$.

3 Existence and Nonexistence

We assume the following conditions:

(H1) $a \in C(T^2)$, $0 \leq a(t, x) \leq c^2/4$ for $(t, x) \in \mathbb{R}^2$, and $\int_{T^2} a(t, x) \, dt \, dx > 0$.
(H2) $f \in C(T^2 \times \mathbb{R}^+, \mathbb{R}^+)$. 
(H3) For all $(t, x) \in T^2$,

$$\lim_{u \to 0^+} \frac{f(t, x, u)}{u} = \infty,$$

where $u \in \mathbb{R}^+$. 
(H4) For all $(t, x) \in T^2$,

$$\lim_{u \to \infty} \frac{f(t, x, u)}{u} = 0,$$

where $u \in \mathbb{R}^+$. 
(H5) For all $(t, x) \in T^2$,

$$\lim_{u \to 0^+} \frac{f(t, x, u)}{u} < \infty, \quad \lim_{u \to \infty} \frac{f(t, x, u)}{u} < \infty,$$

where $u \in \mathbb{R}^+$.

Let $E$ denote the Banach space $C(T^2)$. Hereafter, we simply use $\| \cdot \|$ to denote the norm in Banach $E$ and $\| \cdot \|_p$ to denote the norm in $L^p(T^2)$.

For each $v \in E$, define $v = A \lambda v$ by

$$\begin{cases} 
  u_{tt} - u_{xx} + cu_t + a(t, x)u = \lambda f(t, x, v) & (t, x) \in T^2, \\
  u(t + \pi, x) = u(t, x + \pi) = u(t, x) & (t, x) \in T^2.
\end{cases}$$

(3.1)

Then $A : E \to E$ is well defined, completely continuously, and fixed points of $A \lambda$ are solutions of (1.1) (see, e.g., [7]).

Theorem 3.1. Assume (H1)-(H3) hold. Then there is $\lambda_0 > 0$ such that (1.1) admits a nontrivial nonnegative doubly periodic solution for $0 < \lambda < \lambda_0$.

Proof. By condition (H3), we have

$$\lim_{u \to 0^+} \frac{f(t, x, u)}{u} = \infty,$$
for \( u \in \mathbb{R}^+ \) and \((t, x) \in \mathbb{T}^2\). We can choose \( \delta > 0 \) to get 
\[
 f(t, x, u) > 0 \quad \text{for} \quad 0 < u(t, x) \leq \delta, \quad (t, x) \in \mathbb{T}^2.
\]
Let 
\[
 \lambda_0 = \frac{4}{M M' \int_{\mathbb{T}^2} ds dy} > 0
\]
and 
\[
 M = \sup \{ f(t, x, u) : u(t, x) \leq \delta, \quad (t, x) \in \mathbb{T}^2 \} > 0. \tag{3.2}
\]
We now only consider \( 0 < \lambda < \lambda_0 \). Define a function 
\[
 f_{\min}(t, x, z) : \mathbb{T}^2 \times [0, \delta] \to [0, \infty) \text{ by}
\]
\[
 f_{\min}(t, x, z) = \min\{ f(t, x, u) : u \in \mathbb{R}^+, \quad z \leq u(t, x) \leq \delta, \quad (t, x) \in \mathbb{T}^2 \}.
\]
According to Lemma 2.1, condition (H3) implies 
\[
 \lim_{z \to 0^+} \frac{f_{\min}(t, x, z)}{z} = \infty.
\]
Therefore, for each \( 0 < \lambda < \lambda_0 \), there exists a positive \( \epsilon_1 < \delta \) such that 
\[
 f_{\min}(t, x, \alpha) \geq \frac{\lambda \lambda_1}{\lambda_0} \alpha \quad \text{if} \quad 0 < \alpha \leq \epsilon_1 \quad \text{and} \quad \lambda_1 = \frac{1}{M M' \int_{\mathbb{T}^2} ds dy}.
\]
Now choose an \( \epsilon \) such that \( 0 < \epsilon < \epsilon_1 \). We define a subset \( K \) of \( \mathbb{E} \) by 
\[
 K = \{ u \in \mathbb{E} : \epsilon \leq u(t, x) \leq \delta, \quad \forall (t, x) \in \mathbb{T}^2 \}
\]
for each \( 0 < \lambda < \lambda_0 \). Note that \( \epsilon < \epsilon_1 < \delta \). It is easy to verify that \( K \) is a closed, bounded, convex subset of \( \mathbb{E} \). We show that \( A : K \to K \), in other words, \( u = A v \) for \( v \in K \) we can obtain \( u \in K \).
First, since \( \epsilon \leq v(t, x) \leq \delta \), (3.1) and (3.2), we have 
\[
 u_{tt} - u_{xx} + cu_t + a(t, x) u = \lambda f(t, x, v) \leq \lambda M \quad \forall (t, x) \in \mathbb{T}^2, \tag{3.4}
\]
by (2.4), (2.7) and (3.4), which implies, 
\[
 u(t, x) = \int_{\mathbb{T}^2} G(t - s, x - y) h(s, y) ds dy = \int_{\mathbb{T}^2} G(t - s, x - y) \lambda f(s, y, v) ds dy 
\]
\[
 \leq \lambda M \int_{\mathbb{T}^2} G(t - s, x - y) ds dy 
\]
\[
 \leq \lambda M M' \int_{\mathbb{T}^2} ds dy < \lambda_0 M M' \int_{\mathbb{T}^2} ds dy < \delta.
\]
Finally, by the definition of \( f_{\min} \), we obtain 
\[
 u_{tt} - u_{xx} + cu_t + a(t, x) u = \lambda f(t, x, v) \geq \lambda f_{\min}(t, x, \epsilon).
\]
By the choice of \( \epsilon \) and (3.3), for each \( 0 < \lambda < \lambda_0 \), we have 
\[
 u_{tt} - u_{xx} + cu_t + a(t, x) u \geq \frac{\lambda \lambda_1}{\lambda_0} \epsilon \geq \lambda_1 \epsilon. \tag{3.5}
\]
By (2.4), (2.6) and (3.5), so that
\[ u(t, x) = \int_{\mathbb{T}^2} G(t-s, x-y) h(s, y) ds dy \]
\[ \geq \int_{\mathbb{T}^2} G(t-s, x-y) \lambda_1 \epsilon ds dy \]
\[ \geq \lambda_1 \epsilon \int_{\mathbb{T}^2} ds dy = \epsilon. \]

Hence, \( u \in K \) and \( A_\lambda : K \to K \). By standard methods and Arzelà-Ascoli theorem, the complete continuity of \( A_\lambda : K \to K \) is obvious. So it is omitted. For each \( 0 < \lambda < \lambda_0 \), according to Lemma 2.3, \( A_\lambda \) admits a fixed point in \( K \), which is the desired nontrivial doubly periodic solution of (1.1).

Example 3.2. Consider the following problem:
\[
\begin{align*}
    &u_{tt} - u_{xx} + cu_t + \sin^2(t + x)u = \lambda e^u \\
    &u(t + \pi, x) = u(t, x + \pi) = u(t, x) \quad (t, x) \in \mathbb{T}^2.
\end{align*}
\]
It is clear to see that all conditions of Theorem 3.1 hold.

Theorem 3.3. Assume (H1)-(H4) hold and suppose that, for \( f(t, x, u) \) in (H3),
\[ f(t, x, u) > 0 \quad \text{for} \quad 0 < u(t, x), (t, x) \in \mathbb{T}^2. \]

Then (1.1) admits a nontrivial nonnegative doubly periodic solution for all \( \lambda > 0 \).

Proof. We define a function \( f^{\max} : \mathbb{T}^2 \times [0, \infty) \to [0, \infty) \) by
\[ f^{\max}(t, x, z) = \min \{ f(t, x, u) : u \in \mathbb{R}^+, u(t, x) \leq z \} \quad (t, x) \in \mathbb{T}^2. \]
In consideration of Lemma 2.2, The theorem has
\[ \lim_{u \to \infty} \frac{f(t, x, u)}{u} = 0 \quad u \in \mathbb{R}^+ \text{ and } (t, x) \in \mathbb{T}^2, \]
then
\[ \lim_{z \to \infty} \frac{f^{\max}(t, x, z)}{z} = 0. \]
We can choose a sufficient large \( \delta > 0 \) so that
\[ \frac{f^{\max}(\delta)}{\delta} < \eta, \quad (3.6) \]
where \( \eta > 0 \) satisfying
\[ \lambda \eta \int_{\mathbb{T}^2} ds dy \leq 1. \quad (3.7) \]
With this \( \delta \), we define a function \( f^{\min}(t, x, z) : \mathbb{T}^2 \times [0, \delta] \to [0, \infty) \) by
\[ f^{\min}(t, x, z) = \min \{ f(t, x, u) : u \in \mathbb{R}^+, z \leq u(t, x) \} \quad (t, x) \in \mathbb{T}^2. \]
In view of Lemma 2.1, condition (H3) implies
\[ \lim_{z \to 0^+} \frac{f^{\min}(t, x, z)}{z} = \infty. \]
There exists a positive \( \epsilon_1 < \delta \) such that
\[ f^{\min}(\alpha) \geq \frac{\lambda_1}{\lambda} \alpha \quad (3.8) \]
if \(0 < \alpha \leq \epsilon_1\) and \(\lambda_1 = \frac{1}{2} \int_{\mathbb{T}^2} dsdy\).

Now choose a positive \(\epsilon\) such that \(0 < \epsilon < \epsilon_1 < \delta\). We now define a subset \(K\) of \(E\) by

\[
K = \{ u \in E : \epsilon \leq u(t, x) \leq \delta, \forall (t, x) \in \mathbb{T}^2 \}.
\]

Then \(K\) is a closed, bounded, convex subset of \(E\). We claim that \(u = A_\lambda \) for \(v \in K\), we can have \(u \in K\). First, by the definition of \(f_{\min}\) and (3.6), \(\epsilon \leq v(t, x) \leq \delta\), so we have

\[
\lambda_1 = \frac{1}{2} \int_{\mathbb{T}^2} dsdy.
\]

By (2.4), (2.7), (3.7) and (3.9), which implies,

\[
\lambda \eta \delta \leq \delta.
\]

Finally, by the definition of \(f_{\min}\) and the choice of \(\epsilon, \epsilon_1\), we have

\[
u_{tt} - \nu_{xx} + c\nu_t + a(t, x)\nu = \lambda f(t, x, \nu) \geq \lambda f_{\min}(t, x, \epsilon) \geq \lambda \frac{\lambda_1}{\lambda_1} \epsilon \geq \lambda_1 \epsilon.
\]

In a similar way, we also obtain

\[
u(t, x) \geq \epsilon.
\]

In conclusion, \(u \in K\) and \(A_\lambda : K \to K\). By standard methods and Arzelà-Ascoli theorem, the complete continuity of \(A_\lambda : K \to K\) is obvious. So it is omitted. By Lemma 2.3, \(A_\lambda\) admits a fixed point in \(K\), which is the desired nontrivial doubly periodic solution of (1.1).

**Example 3.4.** Consider the following problem:

\[
\begin{align*}
u_{tt} - \nu_{xx} + c\nu_t + a(t, x)\nu &= \lambda f(t, x, \nu) \leq \lambda C \nu, \\
u(t + \pi, x) &= \nu(t, x + \pi) = \nu(t, x) \quad (t, x) \in \mathbb{T}^2.
\end{align*}
\]

It is clear to see that all conditions of Theorem 3.2 hold.

The following will allow us to establish a nonexistence theorem:

**Theorem 3.5.** Assume (H1)-(H2) and (H5) hold. Then there is \(\lambda_0 > 0\) such that (1.1) admits no nontrivial doubly periodic solution for \(0 < \lambda < \lambda_0\).

**Proof.** By conditions (H1)-(H2) and (H5), there exists a constant \(C > 0\) such that

\[
f(t, x, u) \leq C, \quad u \in \mathbb{R}^+, \quad (t, x) \in \mathbb{T}^2.
\]

Choose \(\lambda_0 > 0\) so that

\[
\lambda_0 C \eta \int_{\mathbb{T}^2} dsdy < 1.
\]

Now assume \(v \in K\) is a nontrivial solution of (1.1). We will prove that this leads to a contradiction if \(0 < \lambda < \lambda_0\). For (3.11), \(0 < \lambda < \lambda_0\) and \((t, x) \in \mathbb{T}^2\)

\[
v_{tt} - v_{xx} + cv_t + a(t, x)v = \lambda f(t, x, v) \leq \lambda C \nu.
\]
By (2.4), (2.7) and (3.12), then
\[ v(t, x) = \int_{T^2} G(t-s, x-y) \lambda f(t, x, v) dsdy \]
\[ \leq \lambda C \|v\| \int_{T^2} G(t-s, x-y) dsdy \]
\[ \leq \lambda C \|v\| \overline{G} \int_{T^2} dsdy < \alpha \|v\|, \]
where \( \alpha = \lambda_0 C \overline{G} \int_{T^2} dsdy \). Thus
\[ \|v\| \leq \alpha \|v\|, \]
which is a contradiction since \( \alpha < 1 \).

**Example 3.6.** Consider the following problem:
\[
\begin{align*}
    u_{tt} - u_{xx} + cu_t + \sin^2(t + x)u &= \lambda e^{-u} \quad (t, x) \in T^2, \\
    u(t + \pi, x) &= u(t, x + \pi) = u(t, x) \quad (t, x) \in T^2.
\end{align*}
\]
It is clear to see that all conditions of Theorem 3.3 hold.

4 Conclusions

Assume that (H1)-(H3) hold, then there exists \( \lambda_0 > 0 \) such that (1.1) admits a nontrivial nonnegative doubly periodic solution for \( 0 < \lambda < \lambda_0 \). Assume (H1)-(H4) hold and suppose that, for \( f(t, x, u) \) in (H3),
\[ f(t, x, u) > 0 \quad \text{for} \quad 0 < u(t, x), \quad (t, x) \in T^2, \]
then (1.1) admits a nontrivial nonnegative doubly periodic solution for all \( \lambda > 0 \). Assume (H1)-(H2) and (H5) hold, then there is \( \lambda_0 > 0 \) such that (1.1) admits no nontrivial doubly periodic solution for \( 0 < \lambda < \lambda_0 \).

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Competing Interests

Author has declared that no competing interests exist.

References


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