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Solutions of Klein-Gordon Equation by the Laplace Decomposition Method and Modified Laplace Decomposition Method

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

In this article, the Laplace decomposition method and Modified Laplace decomposition method have been employed to obtain the exact and approximate solutions of the Klein-Gordon equation with the initial profile. An approximate solution obtained by these methods is in good agreement with the exact solution and shows that these approaches can solve linear and nonlinear problems very effectively and are capable to reduce the size of computational work.

Keywords: *Laplace decomposition method; Modified Laplace decomposition method; Klein-Gordon equation; Adomian polynomials; partial differential equations.*

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1 Introduction

Partial differential equations play a vital role in many different fields such as plasma physics, finance, elastic media, oceanology, meteorology, mathematical biology, geophysics, nonlinear optics, chemical reaction,

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gravitation, and others. Consequently, considerable attention has been given to the solutions of partial differential equations (PDEs) for physical interest. Many methods are available in the literature for the study of PDEs such as the Adomian decomposition method (ADM) [1], Variational iteration method (VIM) [2], Differential transformation method [3], Homotopy perturbation method (HPM) [4], Laplace decomposition method (LDM) [5,6], Modified Laplace decomposition method (MLDM) [7], and so on.

The Klein-Gordon equation (KGE) is a relativistic wave equation, related to the Schrödinger equation, that describes the behavior of spinless particles. It is suitable for the relativistic particle subject to a general Lorentz vector and scalar potential. It has analytical solutions for some physical potentials such as scalar potential, simple harmonic potential, coulombic, and gravitational potential [8]. The Klein-Gordon equation plays a significant role in physics and other sciences such as contemporary physics, radiation theory, solid-state physics, astrophysics, nonlinear optics, cosmology, thermal equilibrium, soliton, quantum mechanics, field theory, harmonic oscillation of clocks in supersonic flow, classical mechanics, and condensed matter physics. The KGE model is an essential phenomenon in soliton, propagation of dislocations in crystals, and the behavior of elementary particles.

Many researchers studied different aspects of the Klein-Gordon equation. F. Yin et al. proposed a spectral collocation method based on the Legendre wavelets method for the solution of 1D and 2D Klein-Gordon equations [9]. In [10] authors used the Sobolev gradient method to obtain an approximate solution of nonlinear KGE. C. Chang and C. Kuo utilize the Lie group method to solve backward two-dimensional KGE [11]. Q. Li and co-authors proposed the Lattice Boltzmann method for the numerical solution of nonlinear KGE [12]. D. Kaya obtained the traveling wave solution of generalized one-dimensional Klein-Gordon equation by using ADM [13]. In [14] authors employed VHPM to obtain the approximate solution of KGE. M. Dehghan and A. Shokri obtained a numerical solution of the nonlinear KGE with the help of radial basis functions [15]. U. Okorie and co-authors used the hyperbolic potential model to solve the D-dimensions Klein-Gordon equation [16].

Finding exact and approximate solutions of the partial differential equations are of major importance and have widespread applications in applied mathematics and other sciences. In this article, the author implements LDM and MLDM to find exact and approximate solutions to KGE. The organization of this article is as follows: Section 1 is about the introduction, the outline of the methods for KGE is given in Section 2. Section 3 contains applications. Section 4 contains graphical presentation and numerical illustration and Section 5 is about the conclusion.

2 Analysis of the Method

Consider nonlinear Klein-Gordon equation with initial conditions

$$u_{tt}(x,t) - u_{xx}(x,t) + cu(x,t) + Nu(x,t) = h(x,y), \tag{2.1}$$

$$u(x,0) = f(x), u_t(x,0) = g(x),$$

where $h(x,t), f(x), and\ g(x)$ are source terms, $Nu(x,t)$ is a nonlinear term, c is a constant.

Taking the Laplace transform of equation (1) w. r. t. t

$$s^2u(x,s) - su(x,0) - u_t(x,0) = L_t\{h(x,y)\} + L_t\{u_{xx}(x,t) - cu(x,t) - Nu(x,t)\},$$
$$u(x,s) = \frac{1}{s}f(x) + \frac{1}{s^2}g(x) + \frac{1}{s^2}L_t\{h(x,y)\} + \frac{1}{s^2}L_t\{u_{xx}(x,t) - cu(x,t) - Nu(x,t)\}.$$

Taking the inverse Laplace transform w. r. t. t

$$u(x,t) = f(x) + tg(x) + L_t^{-1}\left\{\frac{1}{s^2}L_t\{h(x,y)\}\right\}$$
$$+L_t^{-1}\left\{\frac{1}{s^2}L_t\{u_{xx}(x,t) - cu(x,t) - Nu(x,t)\}\right\}. \tag{2.2}$$

Represent solution of the equation (2.1) in an infinite series form to get

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{2.3}$$

The nonlinear term is represented by an infinite series of the so-called Adomian polynomials

$$Nu = \sum_{n=0}^{\infty} \mathcal{A}_n . \tag{2.4}$$

The Adomian polynomials $\mathcal{A}_n$ are generated with the help of the following relation

$$\mathcal{A}_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{\infty} \lambda^k u_k\right)\right]_{\lambda=0}, \quad n \geq 0. \tag{2.5}$$

From equation (2.2)-(2.4) we get

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + tg(x) + L_t^{-1}\left\{\frac{1}{s^2}L_t\{h(x,y)\}\right\} + L_t^{-1}\left\{\frac{1}{s^2}L_t\left\{\sum_{n=0}^{\infty} u_{n_{xx}}(x,t) - c\sum_{n=0}^{\infty} u_n(x,t) - \sum_{n=0}^{\infty} \mathcal{A}_n\right\}\right\}. \tag{2.6}$$

The components of the series are determined by using following recurrence relation

$$u_0(x,t) = f(x) + tg(x) + L_t^{-1}\left\{\frac{1}{s^2}L_t\{h(x,y)\}\right\}, \tag{2.7}$$

$$u_{n+1}(x,t) = L_t^{-1}\left\{\frac{1}{s^2}L_t\left\{u_{n_{xx}}(x,t) - cu_n(x,t) - \mathcal{A}_n\right\}\right\}, n = 0,1,2,\ldots \tag{2.8}$$

In the Modified Laplace decomposition method, we split the term arising from source term and initial conditions as follows

$$f(x) + tg(x) + L_t^{-1}\left\{\frac{1}{s^2}L_t\{h(x,y)\}\right\} = k_1(x,t) + k_2(x,t).$$

Instead of iteration procedure given by equation (2.7) & (2.8), MLDM suggests the following modification

$$\begin{aligned} u_0(x,t) &= k_1(x,t),\\ u_1(x,t) &= k_2(x,t) + L_t^{-1}\left\{\frac{1}{s^2}L_t\left\{u_{0_{xx}}(x,t) - cu_0(x,t) - \mathcal{A}_0\right\}\right\},\\ u_{n+1}(x,t) &= L_t^{-1}\left\{\frac{1}{s^2}L_t\left\{u_{n_{xx}}(x,t) - cu_n(x,t) - \mathcal{A}_n\right\}\right\}, n = 1,2,\ldots \end{aligned} \tag{2.9}$$

The n-term approximate solution is given by

$$\phi_n(x,t) = \sum_{k=0}^{n} u_k(x,t), \quad where \quad k \geq 0. \tag{2.10}$$

Therefore, the exact solution of eq (1) is

$$u(x,t) = \lim_{n\to\infty} \phi_n(x,t). \tag{2.11}$$

3 Applications

Example 1: Consider nonlinear Klein-Gordon equation [17]

$$u_{tt} - u_{xx} - \frac{3}{4}u + \frac{3}{2}u^3 = 0, \tag{3.1}$$

subject to the initial conditions

$$u(x,0) = -\operatorname{sech}(x)\,, u_t(x,0) = \frac{1}{2}\operatorname{sech}(x)\tanh(x).$$

The exact solution of the eq (3.1) is

$$u(x,t) = -\operatorname{sech}(x + \tfrac{1}{2}t). \qquad (3.2)$$

By using MLDM algorithm, components of the series (2.3) are

$$u_0(x,t) = -\operatorname{sech}(x),$$
$$u_1(x,t) = \frac{1}{2}\operatorname{sech}(x)\tanh(x)t + L_t^{-1}\left\{\frac{1}{s^2}L_t\left\{\mathrm{u}_{0_{\mathrm{xx}}}(\mathrm{x},\mathrm{t}) - \frac{3}{4}u_0(x,t) - \mathcal{A}_0\right\}\right\},$$
$$u_{n+1}(x,t) = L_t^{-1}\left\{\frac{1}{s^2}L_t\left\{\mathrm{u}_{n_{\mathrm{xx}}}(\mathrm{x},\mathrm{t}) - \frac{3}{4}u_n(x,t) - \mathcal{A}_n\right\}\right\},$$

where $\mathcal{A}_n$ are the Adomian polynomials of $Nu = u^3$are defined by the following:

$$\mathcal{A}_0 = u_0^3,$$
$$\mathcal{A}_1 = 3u_0^2u_1,$$
$$\mathcal{A}_2 = 6u_0^2u_2 + 6u_0u_1^2,$$
$$\mathcal{A}_3 = 18u_0^2u_3 + 36u_0u_1u_2 + 6u_1^3 \text{ and so on.}$$

Consequently, the first few components of the series are

$$u_0 = -\operatorname{sech}(x),$$
$$u_1 = \frac{\operatorname{sech}(x)\tanh(x)t}{2} + \frac{\left(\operatorname{sech}(x) - 2\operatorname{sech}(x)\tanh^2 x\right)t^2}{8},$$
$$u_2 = -\frac{\operatorname{sech}(x)\tanh^4(x)t^4}{48} + \frac{\left(4\operatorname{sech}(x)\tanh^3(x) + \left(-2\operatorname{sech}^3(x) - 3\operatorname{sech}(x)\right)\tanh(x)\right)t^3}{48} - \frac{\left(-36\operatorname{sech}^3(x) - 10\operatorname{sech}(x)\right)\tanh^2(x)t^4}{384} - \frac{\operatorname{sech}^5(x)t^4}{24} + \frac{7\operatorname{sech}^3(x)t^4}{192} - \frac{\operatorname{sech}(x)t^4}{128}.$$
$$u_3 = -\frac{\operatorname{sech}(x)\tanh(x)t^6}{1440} + \frac{16\operatorname{sech}(x)\tanh^5(x)t^5}{23040} + \frac{\left(296\operatorname{sech}^3(x) - 24\operatorname{sech}(x)\right)\tanh^3(x)t^5}{23040} + \frac{\left(112\operatorname{sech}^5(x) - 258\operatorname{sech}^3(x) + 9\operatorname{sech}(x)\right)\tanh(x))t^5}{23040} + \frac{\left(296\operatorname{sech}^3(x) - 24\operatorname{sech}(x)\right)\tanh^3(x)t^5}{4608} - \frac{\left(3424\operatorname{sech}^5(x) - 1260\operatorname{sech}^3(x) + 42\operatorname{sech}(x)\right)\tanh^2(x)t^6}{46080} - \frac{3\operatorname{sech}^3(x)\tanh^2(x)t^4}{16} + \frac{\left(112\operatorname{sech}^5(x) - 258\operatorname{sech}^3(x) + 9\operatorname{sech}(x)\right)\tanh(x)t^5}{4608} + \frac{\operatorname{sech}^7(x)t^6}{1440} + \frac{29\operatorname{sech}^5(x)t^6}{2880} - \frac{59\operatorname{sech}^3(x)t^6}{7680} + \frac{\operatorname{sech}(x)t^6}{5120}.$$

Then the 4-term series solution can be written as

$$u(x,t) \approx u_0 + u_1 + u_2 + u_3.$$

One soliton solution is obtained for example 1. Approximate solution is compared with exact solution in Table 1 and Fig. 1(c). Only four terms are used in evaluating the approximate solution and we achieved a very good approximation with respect to the exact solution. Also, the error can be made smaller by adding more terms to the decomposition series.

Example 2: Consider linear nonhomogeneous Klein-Gordon equation [18]

$$u_{tt} - u_{xx} + u = 2\cos(x), \qquad (3.3)$$

with initial conditions

$$u(x,0) = \cos(x),\; u_t(x,0) = 1.$$

The LDM leads to the following recurrence relation:

$$u_0 = \cos(x) + t + \cos(x)t^2,$$
$$u_{n+1}(x,t) = L_t^{-1}\left\{\frac{1}{s^2}L_t\{u_{n_{xx}}(x,t) - u_n\}\right\}, n = 0,1,2,\dots .$$

By using above relation, we get the following first few components of the series

$$u_0 = \cos(x) + t + \cos(x)t^2,$$
$$u_1 = -\cos(x)t^2 - \cos(x)\frac{t^4}{6} - \frac{t^3}{3!},$$
$$u_2 = \cos(x)\frac{t^6}{90} + \cos(x)\frac{t^4}{6} + \frac{t^5}{5!},$$
$$u_3 = -\cos(x)\frac{t^8}{2520} - \cos(x)\frac{t^6}{90} - \frac{t^7}{7!}.$$

In the same manner, the remaining components of the series are easily obtained. Substituting all these values in equation (2.10), we obtain

$$\phi_n(x,t) = \cos(x) + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots + \frac{t^{2n+1}}{(2n+1)!}\right).$$

And hence, by equation (2.11), the exact solution of example 2 is $u(x,t) = \cos(x) + \sin(t)$. The result can be verified through substitution.

Example 3: Consider the following linear homogeneous Klein-Gordon equation [19]

$$u_{tt} - u_{xx} - u = 0, \tag{3.4}$$

subject to the initial conditions

$$u(x,0) = 1 + \sin(x),\ \ u_t(x,0) = 0.$$

Proceeding as before, the components of the series are

$$u_0 = 1 + \sin(x),$$
$$u_1 = L_t^{-1}\left\{\frac{1}{s^2}L_t\{u_{0_{xx}}(x,t) - u_0\}\right\} = \frac{t^2}{2!},$$
$$u_2 = L_t^{-1}\left\{\frac{1}{s^2}L_t\{u_{1_{xx}}(x,t) - u_1\}\right\} = \frac{t^4}{4!},$$
$$u_3 = L_t^{-1}\left\{\frac{1}{s^2}L_t\{u_{2_{xx}}(x,t) - u_2\}\right\} = \frac{t^6}{6!},$$
$$u_4 = L_t^{-1}\left\{\frac{1}{s^2}L_t\{u_{3_{xx}}(x,t) - u_3\}\right\} = \frac{t^8}{8!},$$

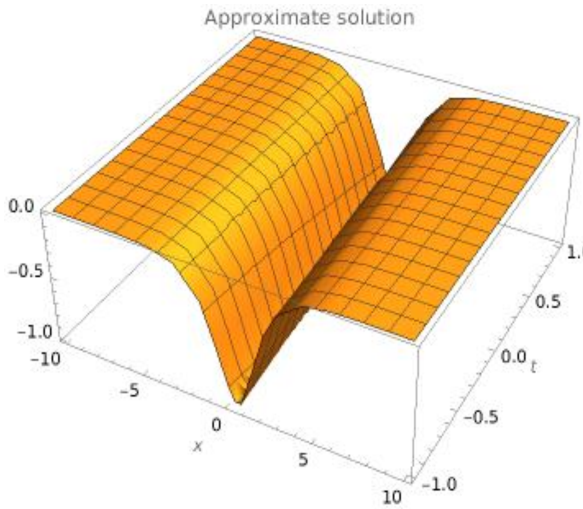
and so on. Hence the n-term approximate solution of example 3 is

$$\phi_n(x,t) = 1 + \sin(x) + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \frac{t^8}{8!} + \dots + \frac{t^{2n}}{2n!}.$$

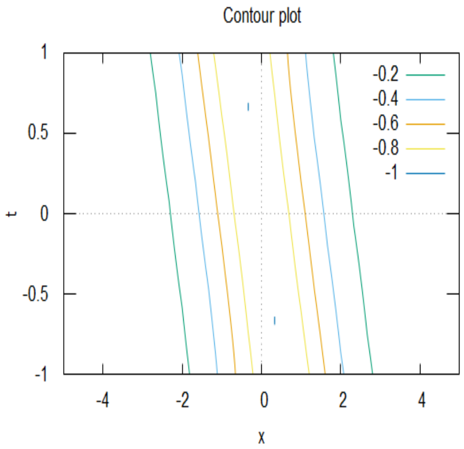
Therefore, by equation (2.11) exact solution of example 3 is

$$u(x,t) = \sin(x) + \cosh(t).$$

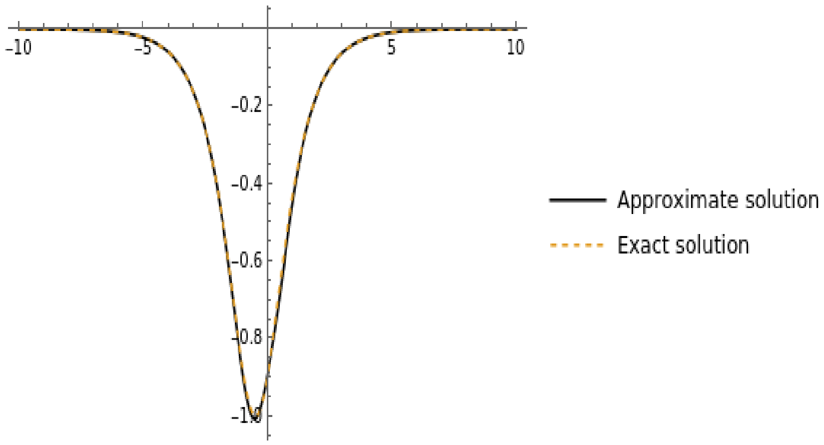
4 Graphical Presentation and Numerical Illustration



(a)



(b)

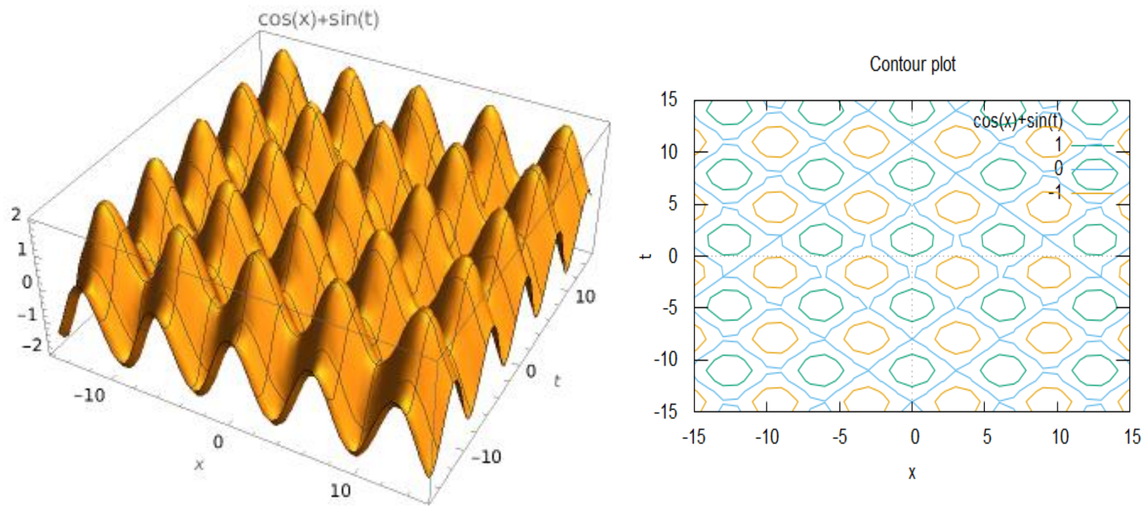


(c)

Fig. 1. (a) 3D and (b) contour plot of the approximate solution of example 1 for $-10 \leq x \leq 10$, $-1 \leq t \leq 1$. (c) Comparison between approximate and exact solutions of example 1 at $t = 1, -10 \leq x \leq 10$.

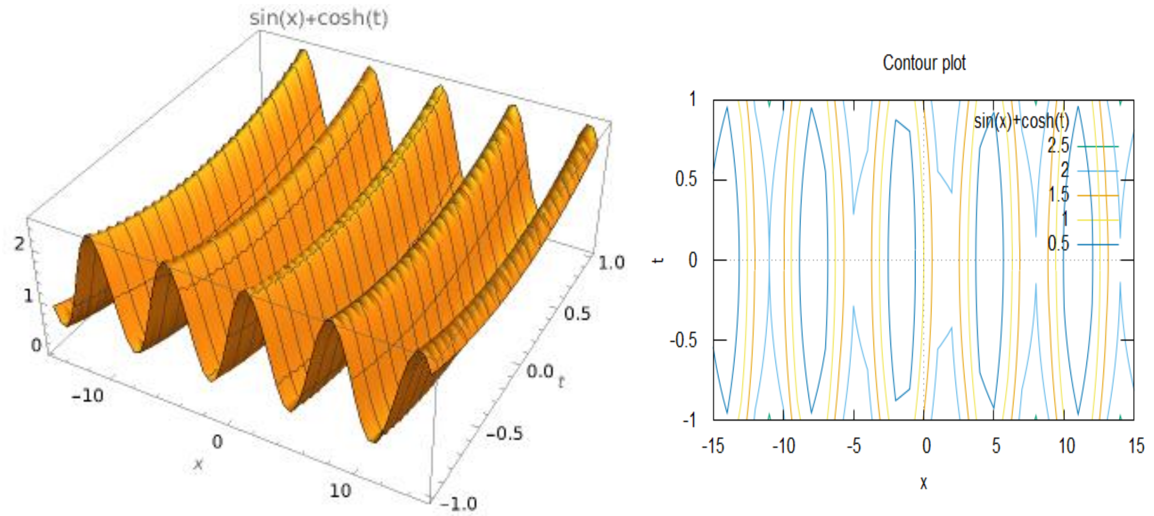
Table 1. The absolute error between exact and approximate solution of example 1 for n=3

x/t	0.1	0.2	0.3	0.4
2	$1.5988E-8$	$2.4840E-6$	$1.2126E-5$	$3.6674E-5$
4	$4.4678E-10$	$6.8930E-9$	$3.3360E-8$	$9.9815E-8$
6	$1.1106E-12$	$1.7217E-11$	$8.4467E-11$	$2.5989E-10$
8	$2.8550E-15$	$5.5584E-14$	$4.2854E-13$	$2.2760E-12$
10	$2.0880E-17$	$1.8845E-15$	$3.0723E-14$	$2.2648E-13$



(a) (b)

Fig. 2. (a) 3D visualization (b) contour plot of the exact solution of example 2 for $-15 \leq x \leq 15, -15 \leq t \leq 15$



(a) (b)

Fig. 3. (a) 3D and (b) contour plot of the exact solution of example 3 for $-15 \leq x \leq 15, -1 \leq t \leq 1$

5 Conclusion

The Laplace decomposition method and MLDM have been successfully implemented for one soliton solution and periodic solutions of the Klein-Gordon equation. Figures are depicting the three-dimensional surfaces and contour plots of the obtained results. The numerical result illustrates the validity and great potential of the proposed methods. The solutions are very rapidly convergent by utilizing these methods. Furthermore, they are not affected by computation round-off errors and it does not require any discretization, linearization, and perturbation. These methodologies can be applied to various types of linear or nonlinear differential equations.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Jebari R, Ghanmi I, Boukricha A. Adomian decomposition method for solving nonlinear heat equation with exponential nonlinearity. I. J. of Math. Analysis. 2013;7(15):725-734.

[2] Huan He J, Hong Wu H. Variational iteration method: New development and applications. Comp. and Math. with Appl. 2007;54(7-8):881-894.

[3] Ayaz F. Solutions of the system of differential equations by differential transform method. Appl. Math. and Comp. 2004;147:547-567.

[4] Biazar J, Eslami M. A new homotopy perturbation method for solving system of partial differential equations. Comp. and Math with Appl. 2011;62(1):225-234.

[5] Khuri S. A Laplace decomposition algorithm applied to a class of nonlinear differential equations. Hindawi Publ. Corp. J. of Appl. Math. 2001;1(4):141-155.

[6] Handibag S, Wayal R. Study of some system of nonlinear partial differential equations by LDM and MLDM. I. J. Sci and Research Publications. 2021;11(6):449-456.

[7] Hussain M, Khan M. Modified Laplace decomposition method. Appl. Math. Sci. 2010;4(36):1769-1783.

[8] Miraboutalebia S. Solutions of Klein-Gordon with Mie-type potential via the Laplace transform. Eur. Phys. J. Plus. 2020;135:16.

[9] Yin F, Tian T, Song J, Zhu M. Spectral methods using Legendre wavelets for nonlinear Klein\Sine-Gordon equations. J. Comp and Appl Math. 2015;275:321-334.

[10] Raza N, Butt A, Javid A. Approximate solution of nonlinear Klein-Gordon equation using Sobolev gradient. Hindawi Publishing C. J. of Function Space; 2016. Article Id 1391594, 7 pages.

[11] Chang C, Kuo C. A Lie group approach for solving backward two-dimensional nonlinear Klein-Gordon equation. Procedia Engineering. 2014;79:590-598.

[12] Li Q, Ji Z, Zheng Z, Liu H. Numerical solution of nonlinear Klein-Gordon equation using Lattice Boltzmann method. Appl. Math. 2011;2:1479-1485.

[13] Kaya D. An implementation of the ADM for generalized one-dimensional Klein-Gordon equation. App. Math. and Comp. 2005;166:426-433.

[14] Yousif M, Mahmood B. Approximate solutions for solving the Klein-Gordon and sine-Gordon equations. J. A. A. Uni. for Basic and Appl. Sci. 2017;22:83-90.

[15] Dehghan M, Shokri A. Numerical solution of the nonlinear Klein-Gordon equation using radial basis functions. J. Comp. and App. Sci. 2009;230(2):400-410.

[16] Okorie U, Ikot A, Edet C, Rampho G, Sever R, Akpan I. Solutions of the Klein-Gordon equation with generalized hyperbolic potential in D-dimensions. J. Phy. Comm. 2019;3:095015.

[17] Zhao X, Zhi H, Yu Y, Zhang H. A new Riccati equation expansion method with symbolic computation to construct new travelling wave solution of nonlinear differential equations. App. Math. and Comp. 2006;172:24-39.

[18] Kulkarni S, Takale K. Application of Adomian decomposition method for solving linear and nonlinear Klein-Gordon equations. I. J. C. Math and Sci. 2015;2:2.

[19] Sayed S. The decomposition method for studying the Klein-Gordon equation. Chaos, Solitons and Fractals. 2003;18:1025-1030.

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